

Custom Hypergraph Categories via Generalized Relations

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Abstract—Process theories combine a graphical language for compositional reasoning with an underlying categorical semantics. They have been successfully applied to fields such as quantum computation, natural language processing, linear dynamical systems and network theory. When investigating a new application, the question arises of how to identify a suitable process theoretic model.

We present a conceptually motivated parameterized framework for the construction of models for process theories. Our framework generalizes the notion of binary relation along four axes of variation, the truth values, a choice of algebraic structure, the ambient mathematical universe and the choice of proof relevance or provability. The resulting categories are preorder-enriched and provide analogues of relational converse and taking graphs of maps. Our constructions are functorial in the parameter choices, establishing mathematical connections between different application domains. We illustrate our techniques by constructing many existing models from the literature, and new models that open up ground for further development.

I. INTRODUCTION

The term “process theory” has recently been introduced [1] to describe compositional theories of abstract processes. These process theories typically consist of a graphical language for reasoning about composite systems, and a categorical semantics tailored to the application domain. This compositional perspective has been incredibly successful in reasoning about questions in quantum computation and quantum foundations. The scope of the process theoretic perspective encompasses many other application domains, including natural language processing [2], signal flow graphs [3], control theory [4], Markov processes [5], electrical circuits [6] and even linear algebra [7].

When considering a new application of the process theoretic approach, the question arises of how to find a suitable categorical setting capturing the phenomena of interest. Dagger compact closed categories are of particular importance as they have an elegant graphical calculus, and many of the examples cited above live in compact closed categories.

We illustrate the process of constructing new dagger compact closed categories with two examples in the theory of human cognition, as developed in [8], [9]. This is an unconventional application area, and therefore highlights clearly the challenges faced when trying to model a new problem domain in a process theoretic manner.

As our first example, we consider how the notion of convexity can be incorporated into a compact closed setting. Convexity is important in mathematical models of cognition, where it is argued that the meaningful concepts should be

closed under forming mixtures. Informally, if we have a space representing animals, then if two points describe dogs, we would expect any points “in between” should also be models of the concept of being a dog.

An algebraic model of convexity is given by the Eilenberg-Moore algebras of the finite distribution monad. These algebras, referred to as convex algebras, are sets equipped with a well behaved operation for forming convex mixtures of elements. Informally, we denote such a convex mixture as

$$\sum_i p_i x_i$$

where the p_i are positive reals summing to one, and the x_i are elements of the algebra. This notation is not intended to imply there are independent addition and scaling operations that can be applied to the individual elements.

The Eilenberg-Moore category of any monad on **Set**, or in fact any regular category, is itself a regular category. Therefore the category of convex algebras is regular and we can form its category of relations, denoted **ConvexRel**. It is well known that the category of relations over a regular category is a dagger compact closed category [10]. Concretely, a convex relation is an ordinary binary relation R which is closed under forming convex mixtures, in the sense that implications of the following form hold

$$R(a_1, b_1) \wedge \dots \wedge R(a_n, b_n) \Rightarrow R\left(\sum_i p_i a_i, \sum_i p_i b_i\right) \quad (1)$$

A **state** of an object A in a monoidal category is a morphism of type $I \rightarrow A$ where I is the monoidal unit. The states in **ConvexRel** are the convex subsets, as we may have hoped. This model was used as the mathematical basis for a compositional model of cognition [11].

As our second example, we return to the mathematics of cognition. It is natural to think about notions of nearness and distance for models of reasoning, a wolf is nearly a dog, a squirrel is closer to being a rat than an elephant. We would therefore like to capture metrics within our model. We now consider how to introduce metrics into a compact closed setting. The construction used in the previous example is not applicable as the various natural categories of metric spaces are not regular. Therefore, we will require a new approach, which will entail a small detour. We begin by introducing the notion of a quantale.

Definition 1 (Quantale). A **quantale** is a join complete partial order Q with a monoid structure (\otimes, k) satisfying the following distributivity axioms, for all $a, b \in Q$ and $A, B \subseteq Q$

$$a \otimes \left[\bigvee B \right] = \bigvee \{a \otimes b \mid b \in B\}$$

$$\left[\bigvee A \right] \otimes b = \bigvee \{a \otimes b \mid a \in A\}$$

A quantale is said to be **commutative** if its monoid structure is commutative.

Example 2. Every locale [12] is a commutative quantale, and in particular any complete chain is a commutative quantale with

$$\bigvee A = \sup A$$

$$a_1 \otimes a_2 = \min(a, b)$$

$$k = \top$$

- The **Boolean quantale** \mathbf{B} is given by the chain $\{0, 1\}$ with its usual ordering
- The **interval quantale** \mathbf{I} is given by the chain $[0, 1] \subseteq \mathbb{R}$ with its usual ordering
- The quantale \mathbf{F} is given by the chain $[0, \infty]$ of extended positive reals with the *reverse* ordering

An important example of a commutative quantale that does not correspond to a locale is the **Lawvere quantale** \mathbf{C} with underlying set the extended positive reals with reverse order and algebraic structure

$$\bigvee A = \inf A$$

$$a_1 \otimes a_2 = a_1 + a_2$$

$$k = 0$$

A binary relation between two sets A and B can be described by its characteristic function

$$A \times B \rightarrow 2$$

where 2 is the two element set of Boolean truth values. We can generalize the notion of binary relation by allowing the truth values to be taken in a suitable choice of quantale Q , as a function of the form

$$A \times B \rightarrow Q$$

We can see this as a potentially infinite matrix of truth values. These binary relations form a category $\mathbf{Rel}(Q)$, with identities and composition given by suitable generalizations of their matrix theoretic analogues. If the quantale of truth values is commutative, $\mathbf{Rel}(Q)$ is in fact dagger compact closed. So we have found another dagger compact closed category, but what has this got to do with metrics? In order to establish the required connection, we note that we can order relations pointwise in the quantale order, as follows:

$$R \subseteq R' \quad \text{iff} \quad \forall a, b. R(a, b) \leq R'(a, b)$$

This order structure makes $\mathbf{Rel}(Q)$ into a poset-enriched symmetric monoidal category. This means we can consider internal

monads, in the sense of formal category theory [13]. These identify important “structured objects” within our categories as follows.

- The internal monads of $\mathbf{Rel}(\mathbf{B})$ are endo-relations such that

$$R(a, a) \quad \text{and} \quad R(a, b) \wedge R(b, c) \Rightarrow R(a, c)$$

That is, they are preorders.

- The internal monads of $\mathbf{Rel}(\mathbf{I})$ are endo-relations where

$$R(a, a) = 1 \quad \text{and} \quad R(a, b) \wedge R(b, c) \leq R(a, c)$$

We can see these as a fuzzy generalization of the notion of a preorder.

- The key example is the internal monads of $\mathbf{Rel}(\mathbf{C})$. These are endo-relations satisfying

$$R(a, a) = 0 \quad \text{and} \quad R(a, b) + R(b, c) \geq R(a, c)$$

If we read the relation R as a distance function, we see that they are *generalized metric spaces* [14].

- The internal monads of $\mathbf{Rel}(\mathbf{F})$ are endo-relations satisfying

$$R(a, a) = 0 \quad \text{and} \quad \max(R(a, b), R(b, c)) \geq R(a, c)$$

Again, if we regard R as a distance function, these can be seen to be *generalized ultrametric spaces*.

So in particular, $\mathbf{Rel}(\mathbf{C})$ gives us a partial order enriched dagger compact closed category in which the internal monads are generalized metric spaces. Such categories of relations have been proposed as a unifying categorical setting for investigating various topological notions, see [15], [16]. Multi-valued relations have also been investigated for compositional models of natural language [17].

To recap, we have constructed two compact closed categories using differing techniques that can be found in the literature:

- By exploiting relations respecting algebraic structure, standard monad and regular category theory provided us with a category where the states are exactly convex subsets.
- Generalizing the notion of relations in a different direction, we produced a category where the internal monads are generalized metric spaces.

So, using rather ad-hoc methods, we have solved two modelling problems using generalizations of binary relations. This prompts several questions:

- How do these constructions relate to each other? In particular, can we simultaneously work with convexity and metrics in an appropriate setting?
- Can they be seen as instances of a general construction?
- Does the notion of binary relation permit further axes of variation, producing additional examples of compact closed categories?
- As the parameters of our constructions vary, can the resulting categories be related? Formally, this is a question of functoriality in a suitable sense.

These questions provide the starting point for our investigations. We also observe that the categories we identified in our examples are both in fact instances of Fong and Kissinger’s hypergraph categories [18]. These are a particularly well behaved class of dagger compact closed categories, and this will be our technical setting for the remainder of the paper.

We summarize our contributions as follows

- We provide parameterized constructions of hypergraph categories of generalized relations and spans in theorems III.7, IV.3 and IV.8.
- We introduce analogues of the notion of converse of a generalized relation, and taking the graph of an underlying morphism. Many further aspects are shown to commute with this important structure.
- In section V the resulting categories are shown to be appropriately order enriched.
- In section VI we show that generalized spans can be functorially mapped to generalized relations in a manner respecting all the important structure.
- In section VII we show that homomorphisms of truth values functorially induce functors between models, preserving all the important structure.
- In section VIII we show that our constructions are functorial in the choices of algebraic structure. We also describe how the algebraic and truth value structures interact, providing connections with notions resource sensitivity in the sense of linear logic.
- In section IX we show that our constructions are also functorial in the choice of ambient topos, with the quantale structure being transferred along a logical functor.
- In theorem X.1 we show that the functors induced by changes of parameters commute with each other.
- Our methods give explicit concrete descriptions of the mathematical objects of interest, suitable for use in applications.
- We provide many examples illustrating the flexibility of our techniques, particularly to the construction of new and existing models of natural language processing and cognition applications.

Related Work

Categories of relations have been studied in the form of allegories [19]. This work is somewhat removed from our approach, the heavy use of the modular law does not directly yield the graphical phenomena of interest. Of more direct relevance is the concept of cartesian bicategory of [20]. Although graphical notation is not used directly in this work, these categories can be seen as close relatives of the hypergraph categories resulting from our constructions. The emphasis in the study of cartesian bicategories was characterization rather than construction of models.

A somewhat syntactic approach to constructing categories with graphical calculi is the use of PROPs [21], [22]. They have recently been used to construct various categorical models relating to control theory [3], [23], [24]. These methods begin with syntax and equations, and freely derive a resulting

category. This style is most effective when the application under consideration has well understood calculational properties. Our approach instead emphasizes the direct construction of models which can then be investigated for their suitability to a given application.

The beautiful work on decorated cospans and correlations of [25], [18], motivated by the program of network theory initiated in [26], is of most direct relevance to our approach. In a precise sense, the decorated correlation construction is completely generic, every hypergraph category is produced by that construction. Our emphasis is different, we do not aim for maximum generality. Instead, our aim is conceptually motivated parameterization. By providing four clearly motivated features that can be adjusted to application needs, we aim for a practical construction with which investigators using process theories can construct new models with desirable features.

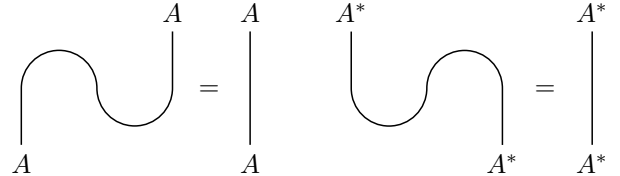
II. MATHEMATICAL BACKGROUND

We will be interested in particular types of symmetric monoidal categories, and will make use of their corresponding graphical languages [27]. Technical background on monoidal categories and general categorical notions can be found in [28]. We will also refer to toposes and their internal languages in places, standard references are [29], [30], [31], [32]. The paper has been written with the intention that it should be readable without any detailed knowledge of topos theory. For such readers, definitions should be read as if they pertain to ordinary sets, functions and predicate logic. In this section we briefly describe some standard mathematical background and conventions.

Definition 1 (Compact Closed Category). An object A in a symmetric monoidal category is said to have dual A^* if there exist unit $\eta : I \rightarrow A^* \otimes A$ and counit $\epsilon : A \otimes A^* \rightarrow I$ morphisms. These morphisms are depicted in the graphical calculus using special notation as



They are required to satisfy the following **snake equations**.



A **compact closed category** is a symmetric monoidal category in which every object has a dual. A compact closed category \mathcal{A} , equipped with an identity on objects involution $(-)^{\dagger} : \mathcal{A}^{op} \rightarrow \mathcal{A}$ coherently with the symmetric monoidal compact closed structure, is referred to as a **dagger compact closed category** [33]. The older term **strongly compact closed category** is also occasionally used.

Example 2. The canonical example of a dagger compact closed category of relevance to the current work is the category **Rel** of sets and binary relations between them. The

symmetric monoidal structure is given by cartesian products of sets, and the dagger by the usual converse of relations. Objects are self-dual, with the unit on a set A given by the relation

$$\eta = \{(*, (a, a)) \mid a \in A\}$$

and the counit is its converse.

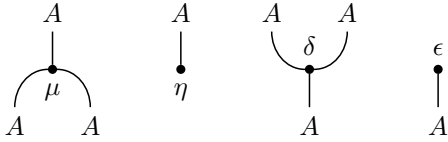
Definition 3 (Hypergraph Category). A **hypergraph category** is a symmetric monoidal category such that every object A carries a commutative monoid structure

$$(\eta : I \rightarrow A, \mu : A \otimes A \rightarrow A)$$

and a cocommutative comonoid structure

$$(\epsilon : A \rightarrow I, \delta : A \rightarrow A \otimes A)$$

We depict these morphisms graphically as follows:



The choice of monoid structure on each object is required to satisfy the following coherence condition with respect to the monoidal structure.

$$\begin{array}{c} A \quad B \quad A \quad B \\ \delta \quad \delta \\ \downarrow \quad \downarrow \\ A \quad B \end{array} = \begin{array}{c} A \otimes B \quad A \otimes B \\ \delta \\ \downarrow \\ A \otimes B \end{array}$$

Here, we overload the use of the symbol δ to avoid cluttering our diagrams with indices or subscripts. We will exploit similar overloading of names in many places in what follows. The monoid structure is also required to satisfy the dual coherence condition. The multiplication and comultiplication must also satisfy the Frobenius axiom

$$\begin{array}{c} A \quad A \\ \mu \quad \delta \\ \downarrow \quad \downarrow \\ A \quad A \end{array} = \begin{array}{c} A \quad A \\ \delta \\ \downarrow \\ A \quad A \end{array} = \begin{array}{c} A \quad A \\ \delta \quad \mu \\ \downarrow \quad \downarrow \\ A \quad A \end{array}$$

and the special axiom

$$\begin{array}{c} A \\ \mu \\ \downarrow \\ A \end{array} = \begin{array}{c} A \\ \delta \\ \downarrow \\ A \end{array}$$

More briefly, a hypergraph category is a symmetric monoidal category with a chosen special commutative Frobenius algebra structure on every object, coherent with the tensor product.

Example 4. The category **Rel** is also an example of a hypergraph category. The cocommutative comonoid is given by the relations

$$\epsilon = \{(a, *) \mid a \in A\} \quad \delta = \{(a, (a, a)) \mid a \in A\}$$

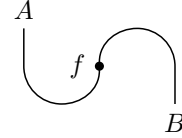
The monoid is the relational converse of the comonoid structure. The induced dagger compact closed structure is exactly that of example 2.

Our interest in hypergraph categories is that they are a particularly pleasant form of dagger compact closed category, as established by the following well known observation.

Proposition 5. Every hypergraph category is a dagger compact closed category, with the cup and cap given by

$$\begin{array}{c} A \quad A \\ \cup \\ A \end{array} = \begin{array}{c} A \quad A \\ \delta \\ \downarrow \\ A \end{array} \quad \begin{array}{c} A \quad A \\ \cap \\ A \end{array} = \begin{array}{c} \epsilon \\ \downarrow \\ A \quad A \\ \mu \end{array}$$

The dagger of a morphism $f : A \rightarrow B$ is given by its transpose



As a final technical point, we will be working with various categories with finite products. Throughout, we will implicitly assume a choice of terminal object and binary products has been given. To reduce clutter, we therefore resist repeating this assumption in the statements of our subsequent theorems.

III. RELATIONS

The aim in this section is to broadly generalize the notion of binary relation between sets, in order to support our motivating examples, and to provide scope for many other variations. We observed, for sets A and B , and quantale Q , that we can consider a function $A \times B \rightarrow Q$ as a relation, with truth values taken in the quantale. For such generalized relations, we define the composition of relations $R : A \rightarrow B$ and $S : B \rightarrow C$ by analogy with the usual composition of relations

$$(S \circ R)(a, c) = \bigvee_b R(a, b) \otimes S(b, c)$$

With this notion of composition, the following relation, with truth values in Q , serves as an identity on set A :

$$1_A(a_1, a_2) = \bigvee \{k \mid a_1 = a_2\}$$

We then observe that all of these definitions actually make sense in the internal language of an arbitrary elementary topos. This leads us to the following definition.

Definition 1 (Q -relation). Let \mathcal{E} be a topos, and (Q, \otimes, k, \bigvee) an internal quantale. A Q -**relation** between \mathcal{E} objects A and B is a \mathcal{E} -morphism of type

$$A \times B \rightarrow Q$$

\mathcal{E} -objects and Q -relations between them form a category $\mathbf{Rel}(Q)$, with identities and composition as described above.

Definition 1 is a first step in the right direction, but in order to capture convexity, as discussed in the introduction, we must find a way of incorporating algebraic structure. If we consider an algebraic signature (Σ, E) with set of operations Σ and equations E , the general form of equation (1), for n -ary operation $\sigma \in \Sigma$, is

$$R(a_1, b_1) \wedge \dots \wedge R(a_n, b_n) \Rightarrow R(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n))$$

We will require throughout that all operation symbols have finite arity, as is conventional in universal algebra.

It is then natural to consider replacing the logical components of this definition with the structure of our chosen quantale. This leads to the definition we require.

Definition 2 (Algebraic Q -relation). Let \mathcal{E} be a topos, and (Q, \otimes, k, \vee) an internal quantale. Let (Σ, E) be an algebraic variety in \mathcal{E} . An **algebraic Q -relation** between (Σ, E) -algebras A and B is a Q -relation between their underlying \mathcal{E} -objects such that for each $\sigma \in \Sigma$ the following axiom holds

$$R(a_1, b_1) \otimes \dots \otimes R(a_n, b_n) \leq R(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n)) \quad (2)$$

(Σ, E) -algebras and algebraic Q -relations form a category $\mathbf{Rel}_{(\Sigma, E)}(Q)$, with identities and composition as for their underlying Q -relations.

There is some subtlety to the interaction between truth values and algebraic structure, we will return to this topic in section VIII. We now continue studying the categorical structure of algebraic Q -relations.

Proposition 3. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a symmetric monoidal category. The symmetric monoidal structure is inherited from the finite products in \mathcal{E} .

We can take the converse of an ordinary binary relation, simply by reversing its arguments. The notion of converse generalizes smoothly to algebraic Q -relations, in a manner that respects all the relevant categorical structure.

Proposition 4. [Converse] Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. There is an identity on objects strict symmetric monoidal functor

$$(-)^\circ : \mathbf{Rel}_{(\Sigma, E)}(Q)^{op} \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

given by reversing arguments:

$$R^\circ(b, a) = R(a, b)$$

For ordinary sets and functions, given a function

$$f : A \rightarrow B$$

we can form a binary relation using the graph of f

$$\{(a, b) \mid f(a) = b\}$$

The next proposition establishes that we can take graphs of morphisms in our underlying category of algebras, in a manner respecting all the relevant categorical structure.

Proposition 5. [Graph] Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. There is an identity on objects strict symmetric monoidal functor

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

defined on morphism $f : A \rightarrow B$ by

$$f_\circ(a, b) = \bigvee \{k \mid f(a) = b\}$$

The symmetric monoidal structure on $\mathbf{Alg}(\Sigma, E)$ is the finite product structure.

The graph functor allows us to lift structures from the underlying category of algebras. The following canonical comonoids are of particular conceptual importance.

Proposition 6. Let \mathcal{E} be a category with finite products. Each object A carries a cocommutative comonoid structure via the canonical morphisms

$$! : A \rightarrow 1 \quad \text{and} \quad \langle 1_A, 1_A \rangle : A \rightarrow A \times A$$

These morphisms satisfy the coherence condition (3).

Finally, we are in a position to establish that our categories of algebraic Q -relations are hypergraph categories.

Theorem III.7. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a hypergraph category. The cocommutative comonoid structure is given by the graphs of the canonical comonoids described in proposition 6, and the monoid structure is given by their converses.

We quickly return to one of the examples discussed in the introduction.

Example 8. The convex algebras discussed in the introduction can be presented by a family of binary operations for forming pairwise convex combinations

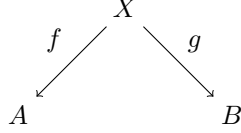
$$+^p \text{ where } p \in (0, 1)$$

satisfying suitable equations. Writing Convex for this signature, we can construct **ConvexRel** as $\mathbf{Rel}_{\text{Convex}}(2)$, where 2 is the two element set.

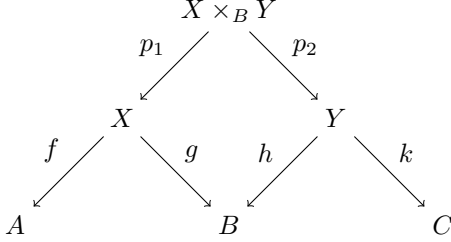
IV. SPANS

Generalizing the truth values, algebraic structure and ambient category has provided three degrees of freedom for describing custom hypergraph categories. Currently we can vary the underlying topos, quantale and choice of algebraic structure. We now investigate a fourth, final direction of variation.

If we consider a span of sets



we can consider an element $x \in X$ as a proof witness relating $f(x)$ and $g(x)$. Two spans are composed by forming the pullback



Recall that pullbacks in **Set** are given explicitly by

$$X \times_B Y = \{(x, y) \mid g(x) = h(y)\}$$

Therefore, a pair (x, y) relates a and c exactly if x relates a to some b and this b is related to c by y . So, at least for the category **Set**, we can think of spans as proof relevant relations. This is the intuition we now pursue, starting by adjusting the notion of Q -relation in definition 1 to the setting of spans.

Definition 1 (Q -span). Let \mathcal{E} be a finitely complete category, and (Q, \otimes, k) an internal monoid. A Q -span of type $A \rightarrow B$ is a quadruple (X, f, g, χ) where

- $(X, f : X \rightarrow A, g : X \rightarrow B)$ is a span in \mathcal{E}
- $\chi : X \rightarrow Q$ is a \mathcal{E} -morphism, referred to as the **characteristic morphism**.

Two Q -spans $(X, f, g, \chi), (Y, h, k, \xi)$ are composed by composing their underlying spans by pullback, and taking the resulting characteristic morphism to be

$$X \times_C Y \xrightarrow{\langle p_1, p_2 \rangle} X \times Y \xrightarrow{\chi \times \xi} Q \times Q \xrightarrow{\otimes} Q$$

where p_1 and p_2 are the pullback projections.

A **morphism of Q -spans** between two Q -spans of type $A \rightarrow B$

$$\alpha : (X_1, f_1, g_1, \chi_1) \rightarrow (X_2, f_2, g_2, \chi_2)$$

is a \mathcal{E} -morphism $\alpha : X_1 \rightarrow X_2$ such that

$$f_1 = f_2 \circ \alpha \quad g_1 = g_2 \circ \alpha \quad \chi_1 = \chi_2 \circ \alpha$$

Remark 2. When discussing Q -spans in the remainder of this paper, we actually intend isomorphism classes of spans with respect to the homomorphisms of definition 1. This convention is common when considering categories of ordinary spans, where composition of spans via pullback is only defined up to isomorphism. All definitions and calculations using representatives will respect this isomorphism structure. These

isomorphism classes of Q -spans form a category $\mathbf{Span}(Q)$. If we write χ_k for the constant morphism

$$\chi_k = A \xrightarrow{1} 1 \xrightarrow{k} Q$$

then the identity at A is given by the Q -span $(A, 1, 1, \chi_k)$.

These spans with configurable truth values provide another construction of hypergraph categories.

Theorem IV.3. Let \mathcal{E} be a finitely complete category, and (Q, \otimes, k) an internal commutative monoid. The category $\mathbf{Span}(Q)$ is a hypergraph category.

We will not dwell on the explicit symmetric monoidal and hypergraph structures claimed in theorem IV.3. Once we have incorporated algebraic structure into our span constructions, the required details can be found in proposition 5 and theorem IV.8.

The key step now is incorporate algebraic structure into the picture, paralleling the ideas of definition 2. In this case, things are slightly more complicated as we have to explicitly administer the proof witnesses in the spans. We also must introduce an ordering on our truth values in order to specify the necessary axiom.

Definition 4. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. For (Σ, E) -algebras A and B , an **algebraic Q -span** is a quadruple (X, f, g, χ) which is a Q -span between the underlying \mathcal{E} -objects satisfying the following axiom.

For every $\sigma \in \Sigma$ if

$$f(x_1) = a_1 \wedge g(x_1) = b_1 \wedge \dots \wedge f(x_n) = a_n \wedge g(x_n) = b_n$$

then there exists x such that

$$f(x) = \sigma(a_1, \dots, a_n) \wedge g(x) = \sigma(b_1, \dots, b_n)$$

and

$$\chi(x_1) \otimes \dots \otimes \chi(x_n) \leq \chi(x)$$

(Σ, E) -algebras and algebraic Q -spans form a category $\mathbf{Span}_{(\Sigma, E)}(Q)$ with identities and composition given as for the underlying Q -spans.

As with the algebraic Q -relations in section III, we obtain a symmetric monoidal category with analogues of relational converse and taking graphs.

Proposition 5. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a symmetric monoidal category. The symmetric monoidal structure is inherited from the finite product structure in \mathcal{E} .

Proposition 6. [Converse] Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. There is an identity on objects strict symmetric monoidal functor

$$(-)^\circ : \mathbf{Span}_{(\Sigma, E)}(Q)^{op} \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q)$$

given by reversing the legs of the underlying span:

$$(X, f, g, \chi)^\circ = (X, g, f, \chi)$$

Proposition 7. [Graph] Let \mathcal{E} be a topos, and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. There is an identity on objects strict symmetric monoidal functor

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q)$$

with the action on morphism $f : A \rightarrow B$ given by

$$f_\circ = (A, 1, f, \chi_k)$$

As before, we can exploit the graph construction and the canonical comonoids of proposition 6 to establish the existence of a hypergraph structure.

Theorem IV.8. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a hypergraph category. The cocommutative comonoid structure is given by the graphs of the canonical comonoids described in proposition 6, and the monoid structure is given by their converses.

This construction presents new modelling possibilities, that can be combined with other features, opening fresh directions for investigation that may not have been immediately apparent.

Example 9. The span construction allows us to construct variations on the models we are already interested in. For example, we can now construct a proof relevant version of the model 8. From a practical perspective, this presents the possibility of models in which we can describe the interaction of cognitive phenomena, and provide quantitative evidence for any relationships that we conclude hold.

Example 10. Instead of using **Set** as our base topos in our models, we could consider using a presheaf topos $[C^{op}, \mathbf{Set}]$ for a small category C . This allows us to construct models using “sets varying with context”, incorporating all the features discussed in the previous examples. In linguistic or cognitive examples, contexts could describe time, the agents involved or the broader setting in which meaning should be interpreted. These context sensitive models present a lot of new expressive potential, and will be investigated in future work.

V. ORDER ENRICHMENT

In order to meaningfully discuss internal monads, we require some 2-categorical structure on our relational constructions. Specifically, we introduce an appropriate ordering on our morphisms. Order enrichment is also important from a practical perspective when modelling real world applications. For example, in natural language applications, we are often interested in phenomena such as ambiguity [34], [35] and lexical entailment [36], and these are best studied from an order theoretic perspective.

Generalizing the situation for ordinary set theoretic binary relations, we introduce an ordering on Q -relations.

Definition 1. Let \mathcal{E} be a topos and (Q, \otimes, k, \vee) an internal quantale. We define a partial order on Q -relations as follows

$$R \subseteq R' \quad \text{iff} \quad \forall a, b. R(a, b) \leq R'(a, b)$$

Algebraic Q -relations are ordered similarly, by comparing their underlying Q -relations.

Theorem V.2. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a partially ordered symmetric monoidal category.

Q -spans can also be ordered, in a manner analogous to that for relations, but explicitly taking into account the proof witnesses.

Definition 3. For topos \mathcal{E} and internal partially ordered monoid Q , we define a preorder on Q -spans as follow.

$$(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$$

if there is a \mathcal{E} -monomorphism $m : X_1 \rightarrow X_2$ such that

$$f_1 = f_2 \circ m \text{ and } g_1 = g_2 \circ m \text{ and } \forall x. \chi_1(x) \leq \chi_2(m(x))$$

Algebraic Q -spans are ordered similarly, by comparing their underlying Q -spans.

Theorem V.4. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a preordered symmetric monoidal category.

The orders are respected by the important converse operation

Proposition 5. Let \mathcal{E} be a topos, and (Σ, E) a variety in \mathcal{E} . Converses respect order structure, in that

- If (Q, \otimes, k, \vee) is an internal quantale, the converse functor of proposition 4 is a partially ordered functor
- If (Q, \otimes, k, \leq) is an internal partially order monoid, the converse functor of proposition 6 is a preordered functor

The order enrichment of Q -relations and Q -spans is crucial for us to be able to consider the internal monads central to the second example of the introduction.

Example 6. The model incorporating metric spaces as internal monads, as discussed in the introduction, can be constructed with base topos **Set**, the empty algebraic signature and using the Lawvere quantale **C** as the choice of truth values.

Now we have both algebraic and order structure available to us within the same construction, we can consider combining the features we are interested in, by making appropriate choices for the parameters used in the construction.

Example 7. We now see that we can combine both the convex and metric features in a single model. With underlying topos **Set**, we take our algebraic structure as in example 8 and our quantale **C** as in example 6. In this case we find the

internal monads are distance measures $d : A \times A \rightarrow [0, \infty]$ satisfying

$$\begin{aligned} d(a, a) &= 0 \\ d(a, b) + d(b, c) &\geq d(a, c) \\ d(a_1, a_2) + d(b_1, b_2) &\geq d(a_1 +^p b_1, a_2 +^p b_2) \end{aligned}$$

These are generalized metric spaces that respect convex structure. The usual metric on \mathbb{R}^n is an example of such a metric.

VI. FROM SPANS TO RELATIONS

We now begin our study of the relationship between the different parameter choices we can take. We start with the simplest case, the binary choice between proof relevance and provability.

The next theorem shows that the orders on relations and spans are compatible, in the sense that we can collapse spans to relations using the join of a quantale to choose optimal truth values, and this mapping is functorial and respects the order structure.

Theorem VI.1. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. There is an identity on objects, strict symmetric monoidal **Preord**-functor*

$$V : \mathbf{Span}_{(\Sigma, E)}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

As we would expect, this extensional collapse of proof witnesses interacts well with the graph and converse operations, and therefore preserves our chosen hypergraph structure on the nose.

Proposition 2. *With the same assumptions, the functor V of theorem VI.1 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{Span}_{(\Sigma, E)}(Q) & \xrightarrow{V} & \mathbf{Rel}_{(\Sigma, E)}(Q) \\ \swarrow (-)_{\circ} & & \nwarrow (-)_{\circ} \\ & \mathbf{Alg}(\Sigma, E) & \\ \mathbf{Span}_{(\Sigma, E)}(Q)^{op} & \xrightarrow{V^{op}} & \mathbf{Rel}_{(\Sigma, E)}(Q)^{op} \\ \downarrow (-)^{\circ} & & \downarrow (-)^{\circ} \\ \mathbf{Span}_{(\Sigma, E)}(Q) & \xrightarrow{V} & \mathbf{Rel}_{(\Sigma, E)}(Q) \end{array}$$

VII. CHANGING TRUTH VALUES

We would expect that homomorphisms between our structures of truth values lead to functorial relationships between models. This all goes through very smoothly, as we now elaborate.

Firstly, for algebraic Q -relations, it is natural to consider internal quantale homomorphisms.

Theorem VII.1. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and $h : Q_1 \rightarrow Q_2$ a morphism of internal commutative*

*quantales. There is an identity on objects, strict symmetric monoidal **Pos**-functor*

$$h^* : \mathbf{Rel}_{(\Sigma, E)}(Q_1) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q_2)$$

The assignment $h \mapsto h^$ is functorial.*

As with the extensional collapse functor of section VI, the induced functor respects graphs and converses, and therefore preserves the hypergraph structure on the nose.

Proposition 2. *With the same assumptions, the functor h^* of theorem VII.1 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{Rel}_{(\Sigma, E)}(Q_1) & \xrightarrow{h^*} & \mathbf{Rel}_{(\Sigma, E)}(Q_2) \\ \swarrow (-)_{\circ} & & \nwarrow (-)_{\circ} \\ & \mathbf{Alg}(\Sigma, E) & \\ \mathbf{Rel}_{(\Sigma, E)}(Q_1)^{op} & \xrightarrow{(h^*)^{op}} & \mathbf{Rel}_{(\Sigma, E)}(Q_2)^{op} \\ \downarrow (-)^{\circ} & & \downarrow (-)^{\circ} \\ \mathbf{Rel}_{(\Sigma, E)}(Q_1) & \xrightarrow{h^*} & \mathbf{Rel}_{(\Sigma, E)}(Q_2) \end{array}$$

In the case of the span constructions, morphisms of partially ordered monoids are the appropriate notion of homomorphism to consider.

Theorem VII.3. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and $h : Q_1 \rightarrow Q_2$ a morphism of internal partially ordered commutative monoids. There is an identity on objects, strict symmetric monoidal **Preord**-functor*

$$h^* : \mathbf{Span}_{(\Sigma, E)}(Q_1) \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q_2)$$

The assignment $h \mapsto h^$ is functorial.*

Again, the induced functor commutes with graphs and converses.

Proposition 4. *With the same assumptions, the functor h^* of theorem VII.3 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{Span}_{(\Sigma, E)}(Q_1) & \xrightarrow{h^*} & \mathbf{Span}_{(\Sigma, E)}(Q_2) \\ \swarrow (-)_{\circ} & & \nwarrow (-)_{\circ} \\ & \mathbf{Alg}(\Sigma, E) & \\ \mathbf{Span}_{(\Sigma, E)}(Q_1)^{op} & \xrightarrow{(h^*)^{op}} & \mathbf{Span}_{(\Sigma, E)}(Q_2)^{op} \\ \downarrow (-)^{\circ} & & \downarrow (-)^{\circ} \\ \mathbf{Span}_{(\Sigma, E)}(Q_1) & \xrightarrow{h^*} & \mathbf{Span}_{(\Sigma, E)}(Q_2) \end{array}$$

Example 5. For any commutative quantale Q there is a partially ordered monoid morphism $1 \rightarrow Q$, induced by the

monoid unit. Here, 1 is the terminal quantale. Therefore there is a strict symmetric monoidal functor

$$\mathbf{Span}_{(\Sigma, E)}(1) \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q)$$

This example motivates our use of partially ordered monoids, rather than simply restricting to the quantales of interest in our primary applications, as the required morphism is not a quantale morphism.

Example 6. There is a quantale morphism $\mathbf{B} \rightarrow \mathbf{C}$ from the Boolean to the Lawvere quantale. The induced functor identifies the ordinary binary relations as living within the category $\mathbf{Rel}(\mathbf{C})$ that we introduced to capture metric spaces as internal monads.

VIII. ALGEBRAIC STRUCTURE

We now investigate the interaction between truth values and algebraic structure. Again, this will lead to functorial relationships between models, but the subject is more delicate than in the previous sections. The essential detail is that inequation (2) is only required to hold for the operations in our signature. It does not directly say anything about derived terms and operations. We will require several definitions in order to make the situation precise.

Definition 1. Let (Σ, E) be an algebraic signature. We say that a term τ over a finite set of variables is

- **Linear** if it uses each variable is used exactly once
- **Affine** if it uses each variable at most once
- **Relevant** if it uses each variable at least once
- **Cartesian** to emphasize that its use of variables is unrestricted

We use the same terminology for the derived operation associated to τ .

Definition 2 (Interpretation). An **interpretation** of signature (Σ_1, E_1) in signature (Σ_2, E_2) is a mapping assigning each $\sigma \in \Sigma_1$ to a derived term of (Σ_2, E_2) of the same arity, such that the equations E_1 can be proved in equational logic from E_2 . We say that an interpretation is linear (affine, relevant, cartesian) if all the derived terms used in the interpretation are linear (affine, relevant, cartesian). We write $\mathbf{Sig}^{\text{lin}}$ ($\mathbf{Sig}^{\text{aff}}$, $\mathbf{Sig}^{\text{rel}}$, $\mathbf{Sig}^{\text{cart}}$) for the corresponding categories with objects signatures and morphisms linear (affine, relevant, cartesian) interpretations.

Definition 3. Let \mathcal{E} be a topos, and (Q, \otimes, k, \leq) an internal partially ordered monoid. We say that Q is

- **Linear** to emphasize that no additional axioms are assumed to hold
- **Affine** if the axiom

$$\forall p, q. p \otimes q \leq p$$

is valid

- **Relevant** if the axiom

$$\forall q. q \leq q \otimes q$$

is valid

- **Cartesian** if it is both affine and relevant

We note the following important special case.

Example 4. A cartesian commutative quantale is a locale.

So in the case where our truth values have a genuine locale structure, everything becomes very well behaved.

Definition 5. Let \mathcal{E} be a topos, and (Q, \otimes, k, \vee) an internal commutative quantale. We say that a Q -relation is

- **Linear** to emphasize that no additional axioms are assumed to hold
- **Affine** if the axiom

$$\forall a_1, a_2, b_1, b_2. R(a_1, b_1) \otimes R(a_2, b_2) \leq R(a_1, b_1)$$

is valid

- **Relevant** if the axiom

$$\forall a, b. R(a, b) \leq R(a, b) \otimes R(a, b)$$

is valid

- **Cartesian** if it is both affine and relevant

We write $\mathbf{Rel}_{(\Sigma, E)}^{\text{lin}}(Q)$, $\mathbf{Rel}_{(\Sigma, E)}^{\text{aff}}(Q)$, $\mathbf{Rel}_{(\Sigma, E)}^{\text{rel}}(Q)$ and $\mathbf{Rel}_{(\Sigma, E)}^{\text{cart}}(Q)$ for the corresponding subcategories of algebraic Q -relations.

Our terminology is derived from that sometimes used for variants of linear logic. If we view truth values as resources, the question is when can these resources be “copied” or “deleted”. We have adopted a naming convention that is slightly redundant, for example $\mathbf{Rel}_{(\Sigma, E)}^{\text{lin}}(Q)$ is the same thing as $\mathbf{Rel}_{(\Sigma, E)}(Q)$. We permit this redundancy in order to allow uniform statements of the subsequent theorems. We begin with important closure properties of our various classes of morphisms.

Proposition 6. *The subcategories of linear (affine, relevant, cartesian) algebraic Q -relations are closed under tensors, converses and the functors induced by quantale homomorphisms. Also, the algebraic Q -relations in the image of the graph functor are all cartesian.*

A straightforward corollary of the closure properties of proposition 6 is

Theorem VIII.7. *For a topos \mathcal{E} , variety (Σ, E) in \mathcal{E} and internal commutative quantale (Q, \otimes, k, \vee) , the categories $\mathbf{Rel}_{(\Sigma, E)}^{\text{lin}}(Q)$, $\mathbf{Rel}_{(\Sigma, E)}^{\text{aff}}(Q)$, $\mathbf{Rel}_{(\Sigma, E)}^{\text{rel}}(Q)$ and $\mathbf{Rel}_{(\Sigma, E)}^{\text{cart}}(Q)$ are sub-hypergraph categories of $\mathbf{Rel}_{(\Sigma, E)}(Q)$.*

Our restricted classes of relations respect the corresponding classes of terms.

Proposition 8. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. For linear (affine, relevant, cartesian) algebraic Q -relation $R : A \rightarrow B$ the axiom*

$$R(a_1, b_1) \otimes \dots \otimes R(a_n, b_n) \leq R(\tau(a_1, \dots, a_n), \tau(b_1, \dots, b_n))$$

holds for every linear (affine, relevant, cartesian) n -ary derived operation τ .

The next proposition is straightforward, it establishes that once our truth values are sufficiently nice, all our relations inherit the same property.

Proposition 9. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. If Q is linear (affine, relevant, cartesian), every morphism in $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is linear (affine, relevant, cartesian).*

In particular, if our quantale is in fact a locale, proposition 9 tells us that everything becomes as straightforward as we might hope.

Finally, we can establish a contravariant functorial relationship between interpretations and functors between models.

Theorem VIII.10. *Let \mathcal{E} be a topos and (Q, \otimes, k, \vee) an internal commutative quantale. Let $i : (\Sigma_1, E_1) \rightarrow (\Sigma_2, E_2)$ be a linear interpretation of signatures. There is a strict monoidal functor*

$$i^* : \mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) \rightarrow \mathbf{Rel}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)$$

The assignment $i \mapsto i^*$ extends to a contravariant functor.

Similar results hold for affine, relevant and cartesian interpretations and relations.

As usual, the induced functors respect the essential graph and converse structure.

Proposition 11. *With the same assumptions, the induced functor i^* of theorem VIII.10 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) & \xrightarrow{i^*} & \mathbf{Rel}_{(\Sigma_1, E_1)}^{\text{lin}}(Q) \\ \uparrow (-)_\circ & & \uparrow (-)_\circ \\ \mathbf{Alg}(\Sigma_2, E_2) & \xrightarrow{i^*} & \mathbf{Alg}(\Sigma_1, E_1) \\ \\ \mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q)^{\text{op}} & \xrightarrow{(i^*)^{\text{op}}} & \mathbf{Rel}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)^{\text{op}} \\ \downarrow (-)^\circ & & \downarrow (-)^\circ \\ \mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) & \xrightarrow{i^*} & \mathbf{Rel}_{(\Sigma_1, E_1)}^{\text{lin}}(Q) \end{array}$$

The bottom functor in these diagrams is the obvious induced functor between categories of algebras. Similar diagrams commute for affine, relevant and cartesian interpretations and relations.

We now introduce similar definitions for the setting of spans, in order to proceed with a similar analysis.

Definition 12. Let \mathcal{E} be a topos, and (Q, \otimes, k, \leq) an internal partially ordered monoid. We say that a Q -span (X, f, g, χ) is

- **Linear** to emphasize that no additional axioms are assumed to hold

- **Affine** if the axiom $\forall x_1, x_2. \chi(x_1) \otimes \chi(x_2) \leq \chi(x_1)$ is valid
- **Relevant** if the axiom $\forall x. \chi(x) \leq \chi(x) \otimes \chi(x)$ is valid
- **Cartesian** if it is both affine and relevant

We write $\mathbf{Span}_{(\Sigma, E)}^{\text{lin}}(Q)$, $\mathbf{Span}_{(\Sigma, E)}^{\text{aff}}(Q)$, $\mathbf{Span}_{(\Sigma, E)}^{\text{rel}}(Q)$ and $\mathbf{Span}_{(\Sigma, E)}^{\text{cart}}(Q)$ for the corresponding subcategories of algebraic Q -spans.

Spans with sufficient structure respect the corresponding types of terms.

Proposition 13. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. For (Σ, E) -algebras A and B , and linear (affine, relevant, cartesian) algebraic Q -span (X, f, g, χ) and n -ary linear (affine, relevant, cartesian) term τ if*

$$f(x_1) = a_1 \wedge g(x_1) = b_1 \wedge \dots \wedge f(x_n) = a_n \wedge g(x_n) = b_n$$

then there exists x such that

$$f(x) = \tau(a_1, \dots, a_n) \wedge g(x) = \tau(b_1, \dots, b_n)$$

and

$$\chi(x_1) \otimes \dots \otimes \chi(x_n) \leq \chi(x)$$

Again, we have good closure of our various classes of morphisms.

Proposition 14. *The subcategories of linear (affine, relevant, cartesian) algebraic Q -spans are closed under tensors, converses and the functors induced by quantale homomorphisms. Also, the algebraic Q -spans in the image of the graph functor are all cartesian.*

As with relations, the closure properties of proposition 14 yield a straightforward corollary about our subcategories of algebraic Q -spans.

Theorem VIII.15. *For a topos \mathcal{E} , variety (Σ, E) in \mathcal{E} and internal partially ordered monoid (Q, \otimes, k, \leq) , the categories $\mathbf{Span}_{(\Sigma, E)}^{\text{lin}}(Q)$, $\mathbf{Span}_{(\Sigma, E)}^{\text{aff}}(Q)$, $\mathbf{Span}_{(\Sigma, E)}^{\text{rel}}(Q)$ and $\mathbf{Span}_{(\Sigma, E)}^{\text{cart}}(Q)$ are sub-hypergraph categories of $\mathbf{Span}_{(\Sigma, E)}(Q)$.*

As with relations, algebraic Q -spans inherit good properties from their underlying quantale.

Proposition 16. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \leq) an internal partially ordered monoid. If Q is linear (affine, relevant, cartesian), every morphism in $\mathbf{Span}_{(\Sigma, E)}(Q)$ is linear (affine, relevant, cartesian).*

Again, we can now establish a contravariant functorial relationship between interpretations and functors between models.

Theorem VIII.17. *Let \mathcal{E} be a topos and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. Let $i : (\Sigma_1, E_1) \rightarrow (\Sigma_2, E_2)$ be a linear interpretation of signatures. There is a strict monoidal functor*

$$i^* : \mathbf{Span}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) \rightarrow \mathbf{Span}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)$$

The assignment $i \mapsto i^*$ extends to a contravariant functor.

Similar results hold for affine, relevant and cartesian interpretations and spans.

The induced functors respect the usual essential structure.

Proposition 18. *For the same assumptions, the induced functor i^* of theorem VIII.17 commutes with graphs and converses. That is, the following diagrams commute*

$$\begin{array}{ccc}
\mathbf{Span}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) & \xrightarrow{i^*} & \mathbf{Span}_{(\Sigma_1, E_1)}^{\text{lin}}(Q) \\
(-)_{\circ} \uparrow & & \uparrow (-)_{\circ} \\
\mathbf{Alg}(\Sigma_2, E_2) & \xrightarrow{i^*} & \mathbf{Alg}(\Sigma_1, E_1) \\
\\
\mathbf{Span}_{(\Sigma_2, E_2)}^{\text{lin}}(Q)^{op} & \xrightarrow{(i^*)^{op}} & \mathbf{Span}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)^{op} \\
(-)^{\circ} \downarrow & & \downarrow (-)^{\circ} \\
\mathbf{Span}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) & \xrightarrow{i^*} & \mathbf{Span}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)
\end{array}$$

The bottom functor in these diagrams is the obvious induced functor between categories of algebras. Similar diagrams commute for affine, relevant and cartesian interpretations and relations.

The extensional collapse functor of section VI also respects our different classes of spans and relations.

Proposition 19. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. The functor of theorem VI.1 maps linear (affine, relevant, cartesian) algebraic Q -spans to linear (affine, relevant, cartesian) algebraic Q -relations.*

We briefly discuss some examples.

Example 20. Let (\emptyset, \emptyset) denote the signature with no operations or equations. For any signature (Σ, E) there is a trivial linear interpretation $(\emptyset, \emptyset) \rightarrow (\Sigma, E)$. We therefore have, for every choice of internal quantale Q , strict symmetric monoidal forgetful functors

$$\begin{aligned}
\mathbf{Rel}_{(\Sigma, E)}(Q) &\rightarrow \mathbf{Rel}_{(\emptyset, \emptyset)}(Q) \\
\mathbf{Span}_{(\Sigma, E)}(Q) &\rightarrow \mathbf{Span}_{(\emptyset, \emptyset)}(Q)
\end{aligned}$$

Example 21. We can present real vector spaces by a signature with a constant element representing the origin, and a family of binary mixing operations, indexed by the scalars involved, satisfying suitable equations. We denote this signature as Linear. There is an interpretation in $\mathbf{Sig}^{\text{lin}}$ of type $\text{Convex} \rightarrow \text{Linear}$. For any commutative quantale Q , this interpretation induces a functor

$$\mathbf{Rel}_{\text{Linear}}(Q) \rightarrow \mathbf{Rel}_{\text{Convex}}(Q)$$

So, as we would expect, we can find the vector spaces in the convex algebras, in a manner respecting all the relevant categorical structure.

Example 22. An affine join semilattice is a set with an associative, commutative, idempotent binary operation. From an information theoretic perspective, we think of convex algebras as describing probabilistic ambiguity. Affine join semilattices can then be thought of as modelling unquantified ambiguity. If we denote the signature for affine join semilattices as Affine there is an interpretation in $\mathbf{Sig}^{\text{lin}}$ of type $\text{Convex} \rightarrow \text{Affine}$ inducing a functor

$$\mathbf{Rel}_{\text{Affine}}(Q) \rightarrow \mathbf{Rel}_{\text{Convex}}(Q)$$

relating these two different models of epistemic phenomena. This exhibits another interesting subcategory of $\mathbf{ConvexRel}$.

IX. CHANGING TOPOS

We now explore the last axis of variation, the topos structure. We would expect that, if \mathcal{E} and \mathcal{F} are elementary toposes, given a suitable functor $L : \mathcal{E} \rightarrow \mathcal{F}$ it would be possible to lift it to a functor between their respective relation and span constructions. Since the definitions of these categories make wide use of the internal language, it should not be surprising that by “suitable” we actually mean that L behaves well with respect to the logical properties of \mathcal{E}, \mathcal{F} .

Definition 1. Given toposes \mathcal{E}, \mathcal{F} , a functor $L : \mathcal{E} \rightarrow \mathcal{F}$ is called *logical* if:

- L preserves products
- L preserves exponentials
- L preserves the subobject classifier.

Logical functors are the right functors to consider, since they preserve the validity of internal formulas: If $\models \phi$ in \mathcal{E} , then $\models \phi$ in \mathcal{F} for every formula ϕ written in the language of first order intuitionistic logic.

To make the following results more readable, we will have to slightly refine our notation, writing $\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q)$ and $\mathbf{Span}_{(\Sigma, E)}^{\mathcal{E}}(Q)$ to explicitly indicate that the constructions are performed on topos \mathcal{E} . If $L : \mathcal{E} \rightarrow \mathcal{F}$ is a logical functor and Q is an internal quantale in \mathcal{E} , then the fact that L preserves models of first order intuitionistic theories implies that LQ is an internal quantale in \mathcal{F} . It makes sense, then, to consider how $\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q)$ and $\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)$ are related. The main result of the section is the following:

Theorem IX.2. *Let \mathcal{E}, \mathcal{F} be toposes, and $L : \mathcal{E} \rightarrow \mathcal{F}$ be a logical functor. Let (Q, \otimes, k, \vee) be an internal commutative quantale in \mathcal{E} and (Σ, E) be a signature. There is a symmetric monoidal functor*

$$L^* : \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)$$

The assignment $L \mapsto L^*$ is functorial.

As in the previous cases, graph and converse functors are preserved.

Proposition 3. *With the same assumptions, the induced functor L^* of theorem IX.2 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc}
\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) & \xrightarrow{L^*} & \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ) \\
(-)_{\circ} \uparrow & & \uparrow (-)_{\circ} \\
\mathbf{Alg}^{\mathcal{E}}(\Sigma, E) & \xrightarrow{L} & \mathbf{Alg}^{\mathcal{F}}(\Sigma, E) \\
\\
\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q)^{op} & \xrightarrow{(L^*)^{op}} & \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)^{op} \\
(-)^{\circ} \downarrow & & \downarrow (-)^{\circ} \\
\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) & \xrightarrow{L^*} & \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)
\end{array}$$

As with relations, morphisms between toposes extend functorially to morphisms between spans.

Theorem IX.4. *Let \mathcal{E}, \mathcal{F} be toposes, and $L : \mathcal{E} \rightarrow \mathcal{F}$ be a logical functor. Let (Q, \otimes, k, \leq) be an internal partially ordered commutative monoid in \mathcal{E} and (Σ, E) be a signature. There is a symmetric monoidal functor*

$$L^* : \mathbf{Span}_{(\Sigma, E)}^{\mathcal{E}}(Q) \rightarrow \mathbf{Span}_{(\Sigma, E)}^{\mathcal{F}}(LQ)$$

The assignment $L \mapsto L^*$ is functorial.

The essential structure is again respected by the induced functors.

Proposition 5. *With the same assumptions, the induced functor L^* of theorem IX.4 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc}
\mathbf{Span}_{(\Sigma, E)}^{\mathcal{E}}(Q) & \xrightarrow{L^*} & \mathbf{Span}_{(\Sigma, E)}^{\mathcal{F}}(LQ) \\
(-)_{\circ} \uparrow & & \uparrow (-)_{\circ} \\
\mathbf{Alg}^{\mathcal{E}}(\Sigma, E) & \xrightarrow{L} & \mathbf{Alg}^{\mathcal{F}}(\Sigma, E) \\
\\
\mathbf{Span}_{(\Sigma, E)}^{\mathcal{E}}(Q)^{op} & \xrightarrow{(L^*)^{op}} & \mathbf{Span}_{(\Sigma, E)}^{\mathcal{F}}(LQ)^{op} \\
(-)^{\circ} \downarrow & & \downarrow (-)^{\circ} \\
\mathbf{Span}_{(\Sigma, E)}^{\mathcal{E}}(Q) & \xrightarrow{L^*} & \mathbf{Span}_{(\Sigma, E)}^{\mathcal{F}}(LQ)
\end{array}$$

Example 6. Given any category \mathcal{C} we can form a corresponding *presheaf category*, having representable functors from \mathcal{C} to \mathbf{Set} as objects and natural transformations between them as morphisms. Presheaves constitute one of the most important examples of toposes, and it makes sense to ask how Theorems IX.2, IX.4 behave in these circumstances.

In general, given arbitrary categories \mathcal{C}, \mathcal{D} it is difficult to say when a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ lifts to a logical functor between the corresponding presheaves. Nevertheless, the following result holds: If \mathcal{C}, \mathcal{D} are groupoids (categories in which every arrow is an isomorphism), then any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ lifts to a logical functor $\bar{F} : \mathcal{C} \rightarrow \mathcal{D}$. This is because truth values in presheaf toposes are defined in terms

of *sieves* (subfunctors of the homset functor) and these sieves trivialize when the only arrows at our disposal are isos. This in turn trivializes the structure of truth values in the presheaf itself, that ends up to be defined pointwise from \mathbf{Set} .

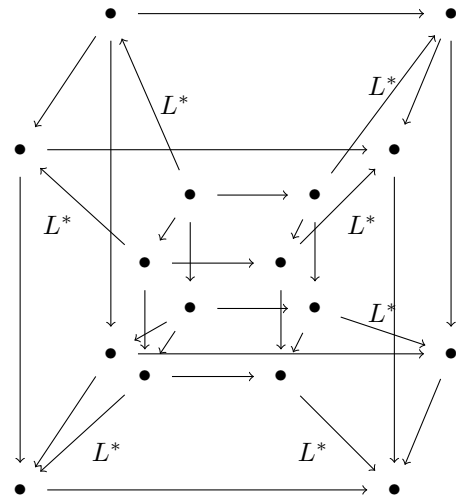
Theorems IX.2, IX.4 then ensure that \bar{F} can be lifted to the relational and span structures built on $\mathbf{Set}^{\mathcal{C}}$ and $\mathbf{Set}^{\mathcal{D}}$, respectively.

Example 7. If \mathcal{E} is a topos, and $f : I \rightarrow J$ is a morphism of \mathcal{E} , then pulling back along f induces a logical functor $F : \mathcal{E}/J \rightarrow \mathcal{E}/I$. Theorem IX.2 guarantees the existence of a functor $F^* : \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}/J}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}/I}(FQ)$. In particular, this means that there is always a functor $F^* : \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}/I}(FQ)$, where \mathcal{E}/I is any slice topos of \mathcal{E} .

X. INDEPENDENCE OF THE AXES OF VARIATION

Finally, we establish that our various induced functors between models are independent, in that they all commute with each other. Unfortunately, the commutativity of the functors induced by interpretations between algebras, order structure and quantale morphisms with L^* will hold only up to isomorphism. This depends intrinsically on the definition of logical functor, that is, in turn, defined to preserve validity of formulas in the internal language only up to natural isomorphism.

Theorem X.1. *Let \mathcal{E} be a topos, $h : Q_1 \rightarrow Q_2$ a morphism of internal commutative quantales, $i : (\Sigma_1, E_1) \rightarrow (\Sigma_2, E_2)$ a linear interpretation and $L : \mathcal{E} \rightarrow \mathcal{F}$ a logical functor. For the induced functors of theorems VII.1, VII.3, VIII.10, VIII.17, IX.2 and IX.4, the following diagram commutes (be aware that in the hypercube below commutative squares involving L^* only commute up to isomorphism. Other squares commute up to equality):*



Where the inner cube is

$$\begin{array}{ccc}
\text{Span}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{E}}(Q_1) & \xrightarrow{i^*} & \text{Span}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{E}}(Q_1) \\
\downarrow h^* & & \downarrow h^* \\
\text{Span}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{E}}(Q_2) & \xrightarrow{i^*} & \text{Span}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{E}}(Q_2) \\
\downarrow & & \downarrow \\
\text{Rel}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{E}}(Q_1) & \xrightarrow{i^*} & \text{Rel}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{E}}(Q_1) \\
\downarrow h^* & & \downarrow h^* \\
\text{Rel}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{E}}(Q_2) & \xrightarrow{i^*} & \text{Rel}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{E}}(Q_2)
\end{array}$$

and the outer cube is

$$\begin{array}{ccc}
\text{Span}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{F}}(LQ_1) & \xrightarrow{i^*} & \text{Span}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{F}}(LQ_1) \\
\downarrow (Lh)^* & & \downarrow (Lh)^* \\
\text{Span}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{F}}(LQ_2) & \xrightarrow{i^*} & \text{Span}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{F}}(LQ_2) \\
\downarrow & & \downarrow \\
\text{Rel}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{F}}(LQ_1) & \xrightarrow{i^*} & \text{Rel}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{F}}(LQ_1) \\
\downarrow (Lh)^* & & \downarrow (Lh)^* \\
\text{Rel}_{(\Sigma_2, E_2)}^{\text{lin}, \mathcal{F}}(LQ_2) & \xrightarrow{i^*} & \text{Rel}_{(\Sigma_1, E_1)}^{\text{lin}, \mathcal{F}}(LQ_2)
\end{array}$$

In both cases the vertical arrows are the functors of theorem VI.1. Similar diagrams commute for affine, relevant and cartesian interpretations, relations and spans.

XI. CONCLUSION

We have developed a parameterized scheme for constructing hypergraph categories, by generalizing the notion of binary relation along four axes of variation:

- The ambient mathematical background via the choice of underlying category
- The truth values via a choice of internal quantale
- The choice of algebraic structure
- The choice between proof relevance and provability

This construction provides a conceptually motivated approach for producing models of process theories when investigating new applications. Many existing examples in the literature are covered by our approach, including examples used for linguistics, cognition, linear dynamical systems and non-deterministic computation.

We showed that the resulting categories are preorder enriched, providing more flexible modelling possibilities. It was also established that varying each of the parameters is functorial, preserving all the important hypergraph and order structure. In the case of the algebraic structure, this functoriality exhibited an interesting relationship between algebra and resource sensitivity in the sense of linear logic. Our constructions were also shown to have well behaved functorial

analogues of the notions of taking the converse of a relations, and taking the graph of a map to construct a new relation.

Interestingly, our framework points to new models in which features can be combined. This was a key objective of this direction of research. For example the model incorporating both convexity and metrics of example 7, the proof relevant models of cognition of example 9 and the possibility of incorporating contexts as discussed in example 10. The application of these constructions to models of cognition and natural language will be explored in forthcoming work.

In order to gain a strict composition operation, in section IV we used isomorphism classes of spans, and then introduced an analogue of the usual order structure for relations in section V. If we use spans, rather than their equivalence classes, they should form a symmetric monoidal bicategory, sacrificing strict composition for a richer 2-cell structure. This is of practical interest as internal monads have been important in our examples. The internal monads in categories of spans correspond to internal categories [37], which would open up further interesting possibilities. Some related work on bicategorical aspects of the decorated cospan construction appears in [38]. In fact, the resulting categories should be an appropriate bicategorical generalization of a hypergraph category. Such bicategorical aspects are left to later work.

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APPENDIX

We outline proofs of the key results.

Proposition 3. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a symmetric monoidal category. The symmetric monoidal structure is inherited from the finite products in \mathcal{E} .*

Proof. The proof that $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a category follows from the unit and associative properties of quantale multiplication. We must first confirm that identities respect algebraic structure. For $\sigma \in \Sigma$

$$\begin{aligned}
 1_A(a_1, a'_1) \otimes \dots \otimes 1_A(a_n, a'_n) &= \\
 &= \left[\bigvee \{k \mid a_1 = a'_1\} \right] \otimes \dots \otimes \left[\bigvee \{k \mid a_n = a'_n\} \right] \\
 &= \bigvee \{k \mid (a_1 = a'_1) \wedge \dots \wedge (a_n = a'_n)\} \\
 &\leq \bigvee \{k \mid \sigma(a_1, \dots, a_n) = \sigma(a'_1, \dots, a'_n)\} \\
 &= 1_A(\sigma(a_1, \dots, a_n), \sigma(a'_1, \dots, a'_n))
 \end{aligned}$$

We do the same for compositions

$$\begin{aligned}
 (S \circ R)(a_1, c_1) \otimes \dots \otimes (S \circ R)(a_n, c_n) &= \\
 &= \left[\bigvee_{b_1} R(a_1, b_1) \otimes S(b_1, c_1) \right] \otimes \dots \\
 &\quad \dots \otimes \left[\bigvee_{b_n} R(a_n, b_n) \otimes S(b_n, c_n) \right] \\
 &= \bigvee_{b_1, \dots, b_n} R(a_1, b_1) \otimes S(b_1, c_1) \otimes \dots \otimes R(a_n, b_n) \otimes S(b_n, c_n) \\
 &= \bigvee_{b_1, \dots, b_n} [R(a_1, b_1) \otimes \dots \otimes R(a_n, b_n)] \otimes \\
 &\quad \otimes [S(b_1, c_1) \otimes \dots \otimes S(b_n, c_n)] \\
 &\leq \bigvee_{b_1, \dots, b_n} R(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n)) \otimes \\
 &\quad \otimes S(\sigma(b_1, \dots, b_n), \sigma(c_1, \dots, c_n)) \\
 &\leq \bigvee_b R(\sigma(a_1, \dots, a_n), b) \otimes S(b, \sigma(c_1, \dots, c_n)) \\
 &= (S \circ R)(\sigma(a_1, \dots, a_n), \sigma(c_1, \dots, c_n))
 \end{aligned}$$

Now we check explicitly the associativity of composition. For relations R, S, T , we have

$$\begin{aligned}
 (T \circ (S \circ R))(a, d) &= \bigvee_c (S \circ R)(a, c) \otimes T(c, d) \\
 &= \bigvee_c \left[\bigvee_d R(a, b) \otimes S(b, c) \right] \otimes T(c, d) \\
 &= \bigvee_d R(a, b) \otimes \left[\bigvee_c S(b, c) \otimes T(c, d) \right] \\
 &= \bigvee_d R(a, b) \otimes (T \circ S)(b, d) \\
 &= ((T \circ S) \circ R)(a, d)
 \end{aligned}$$

We also check the right identity law.

$$(R \circ 1_A)(a, b) = \bigvee_{a'} 1_A(a, a') \otimes R(a', b)$$

$$\begin{aligned}
&= \bigvee_{a'} \left[\bigvee \{k \mid a = a'\} \right] \otimes R(a', b) \\
&= \bigvee_{a'} \bigvee \{k \otimes R(a', b) \mid a = a'\} \\
&= \bigvee_{a'} \{R(a, b)\} \\
&= R(a, b)
\end{aligned}$$

The left identity law proof is similar.

Now we prove this category is symmetric monoidal. We define the monoidal unit to be the terminal algebra. On objects, the tensor takes products of algebras. We define the action on morphisms pointwise as

$$(R \otimes R')(a, a', b, b') = R(a, b) \otimes R'(a', b')$$

We first confirm $R \otimes R'$ respects the algebraic structure. For $\sigma \in \Sigma$ with arity n

$$\begin{aligned}
&(R \otimes R')(a_1, a'_1, b_1, b'_1) \otimes \dots \otimes (R \otimes R')(a_n, a'_n, b_n, b'_n) = \\
&= R(a_1, b_1) \otimes R'(a'_1, b'_1) \otimes \dots \otimes R(a_n, b_n) \otimes R'(a'_n, b'_n) \\
&= [R(a_1, b_1) \otimes \dots \otimes R(a_n, b_n)] \otimes [R'(a'_1, b'_1) \otimes \dots \otimes R'(a'_n, b'_n)] \\
&\leq R(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n)) \otimes \\
&\quad \otimes R'(\sigma(a'_1, \dots, a'_n), \sigma(b'_1, \dots, b'_n)) \\
&= (R \otimes R')((\sigma(a_1, \dots, a_n), \sigma(a'_1, \dots, a'_n)), \\
&\quad (\sigma(b_1, \dots, b_n), \sigma(b'_1, \dots, b'_n))) \\
&= (R \otimes R')(\sigma((a_1, a'_1), \dots, (a_n, a'_n)), \sigma((b_1, b'_1), \dots, (b_n, b'_n)))
\end{aligned}$$

Then, we show that the tensor is bifunctorial. Identities are preserved:

$$\begin{aligned}
&(1_{A_1} \otimes 1_{A_2})(a_1, a_2, a'_1, a'_2) = 1_{A_1}(a_1, a'_1) \otimes 1_{A_2}(a_2, a'_2) \\
&= \bigvee \{k \mid a_1 = a'_1\} \otimes \bigvee \{k \mid a_2 = a'_2\} \\
&= \bigvee \{k \mid (a_1 = a'_1) \wedge (a_2 = a'_2)\} \\
&= \bigvee \{k \mid (a_1, a_2) = (a'_1, a'_2)\} \\
&= 1_{A_1 \otimes A_2}(a_1, a_2, a'_1, a'_2)
\end{aligned}$$

For composition,

$$\begin{aligned}
&[(S_1 \otimes S_2) \circ (R_1 \otimes R_2)](a_1, a_2, c_1, c_2) = \\
&= \bigvee_{b_1, b_2} (R_1 \otimes R_2)(a_1, a_2, b_1, b_2) \otimes (S_1 \otimes S_2)(b_1, b_2, c_1, c_2) \\
&= \bigvee_{b_1, b_2} R_1(a_1, b_1) \otimes R_2(a_2, b_2) \otimes S_1(b_1, c_1) \otimes S_2(b_2, c_2) \\
&= \bigvee_{b_1, b_2} R_1(a_1, b_1) \otimes S_1(b_1, c_1) \otimes R_2(a_2, b_2) \otimes S_2(b_2, c_2) \\
&= \left[\bigvee_{b_1} R_1(a_1, b_1) \otimes S_1(b_1, c_1) \right] \otimes \\
&\quad \otimes \left[\bigvee_{b_2} R_2(a_2, b_2) \otimes S_2(b_2, c_2) \right] \\
&= (S_1 \circ R_1)(a_1, c_1) \otimes (S_2 \circ R_2)(a_2, c_2) \\
&= [(S_1 \circ R_1) \otimes (S_2 \circ R_2)](a_1, a_2, c_1, c_2)
\end{aligned}$$

We consider \mathcal{E} as a symmetric monoidal category with respect to our choice of binary products and terminal object. We then take the graphs (see proposition 5 for the definition) of the corresponding left and right unitors, associator and symmetry as the corresponding structure in $\mathbf{Rel}(Q)$.

We must confirm that these coherence morphisms are natural in their parameters. The proofs are all similar, we check the associator explicitly

$$\begin{aligned}
&R \otimes (S \otimes T) \circ \alpha_{A, B, C} = \\
&= \bigvee_{x, y, z} \alpha_{A, B, C}(((a, b), c), (x, (y, z))) \otimes \\
&\quad \otimes [R \otimes (S \otimes T)]((x, (y, z)), (a', (b', c'))) \\
&= \bigvee_{x, y, z} \left[\bigvee \{k \mid (a = x) \wedge (b = y) \wedge (c = z)\} \right] \otimes \\
&\quad \otimes R(x, a') \otimes S(y, b') \otimes T(z, c') \\
&= \bigvee_{x, y, z} \bigvee \{R(x, a') \otimes S(y, b') \otimes T(z, c') \mid \\
&\quad \mid (a = x) \wedge (b = y) \wedge (c = z)\} \\
&= R(a, a') \otimes S(b, b') \otimes T(c, c') \\
&= \bigvee_{x, y, z} \bigvee \{R(a, x) \otimes S(b, y) \otimes T(c, z) \mid \\
&\quad \mid (x = a') \wedge (y = b') \wedge (z = c')\} \\
&= \bigvee_{x, y, z} R(a, x) \otimes S(b, y) \otimes T(c, z) \otimes \\
&\quad \otimes \left[\bigvee \{k \mid x = a' \wedge y = b' \wedge z = c'\} \right] \\
&= \bigvee_{x, y, z} [(R \otimes S) \otimes T](((a, b), c), ((x, y), z)) \otimes \\
&\quad \otimes \alpha_{A', B', C'}(((x, y), z), (a', (b', c'))) \\
&= \alpha_{A', B', C'} \circ (R \otimes S) \otimes T
\end{aligned}$$

These are isomorphisms by functoriality of graphs (see proposition 5 for the proof). Their inverses are given by their converses, as established in proposition 4.

Moreover, taking graphs commutes with our choice of products in \mathcal{E} in the sense that

$$(f \times g) \circ = f \circ \otimes g \circ$$

To check this we reason as follows:

$$\begin{aligned}
&(f \times g) \circ ((a, a'), (b, b')) = \bigvee \{k \mid (b, b') = (f \times g)(a, a')\} \\
&= \bigvee \{k \mid (b = f(a)) \wedge (b' = g(a'))\} \\
&= \left[\bigvee \{k \mid b = f(a)\} \right] \otimes \left[\bigvee \{k \mid b' = g(a')\} \right] \\
&= f \circ (a, b) \otimes g \circ (a', b')
\end{aligned}$$

This guarantees that the triangle and pentagon equations hold as the same equations hold for the cartesian monoidal structure in \mathcal{E} . The coherence conditions for the symmetry follow similarly. \square

Proposition 5. [Graph] Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \bigvee) an internal commutative quantale.

There is an identity on objects strict symmetric monoidal functor

$$(-)_{\circ} : \mathbf{Alg}(\Sigma, E) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

defined on morphism $f : A \rightarrow B$ by

$$f_{\circ}(a, b) = \bigvee \{k \mid f(a) = b\}$$

The symmetric monoidal structure on $\mathbf{Alg}(\Sigma, E)$ is the finite product structure.

Proof. First of all we have to check that the resulting relation respects the algebraic structure. For $\sigma \in \Sigma$

$$\begin{aligned} f_{\circ}(a_1, b_1) \otimes \dots \otimes f_{\circ}(a_n, b_n) &= \\ &= \left[\bigvee \{k \mid f(a_1) = b_1\} \right] \otimes \dots \\ &\quad \dots \otimes \left[\bigvee \{k \mid f(a_n) = b_n\} \right] \\ &= \bigvee \{k \mid f(a_1) = b_1 \wedge \dots \wedge f(a_n) = b_n\} \\ &\leq \bigvee \{k \mid \sigma(f(a_1), \dots, f(a_n)) = \sigma(b_1, \dots, b_n)\} \\ &= \bigvee \{k \mid f(\sigma(a_1, \dots, a_n)) = \sigma(b_1, \dots, b_n)\} \\ &= f_{\circ}(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n)) \end{aligned}$$

The we confirm this is functorial with respect to identities

$$\begin{aligned} 1_{A_{\circ}}(a_1, a_2) &= \bigvee \{k \mid 1_A(a_1) = a_2\} \\ &= \bigvee \{k \mid a_1 = a_2\} \\ &= 1_A(a_1, a_2) \end{aligned}$$

For functoriality with respect to composition,

$$\begin{aligned} (g_{\circ} \circ f_{\circ})(a, c) &= \bigvee_b f_{\circ}(a, b) \otimes g_{\circ}(b, c) \\ &= \bigvee_b \left[\bigvee \{k \mid f(a) = b\} \right] \otimes \left[\bigvee \{k \mid g(b) = c\} \right] \\ &= \bigvee \{k \mid g(f(a)) = c\} \\ &= (g \circ f)_{\circ}(a, c) \end{aligned}$$

Finally we prove the preservation of the monoidal structure:

$$\begin{aligned} (f \times g)_{\circ}((a, a'), (b, b')) &= \bigvee \{k \mid (b, b') = (f \times g)(a, a')\} \\ &= \bigvee \{k \mid b = f(a) \wedge b' = g(a')\} \\ &= \left[\bigvee \{k \mid b = f(a)\} \right] \otimes \left[\bigvee \{k \mid b' = g(a')\} \right] \\ &= f_{\circ}(a, b) \otimes g_{\circ}(a', b') \\ &= (f_{\circ} \otimes g_{\circ})((a, a'), (b, b')) \end{aligned}$$

Proposition 4. [Converse] Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \bigvee) an internal commutative quantale. There is an identity on objects strict symmetric monoidal functor

$$(-)^{\circ} : \mathbf{Rel}_{(\Sigma, E)}(Q)^{op} \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

given by reversing arguments:

$$R^{\circ}(b, a) = R(a, b)$$

Proof. We first check that taking the converse gives a well defined relation,. For $\sigma \in \Sigma$ of arity n , we reason

$$\begin{aligned} R^{\circ}(a_1, b_1) \otimes \dots \otimes R^{\circ}(a_n, b_n) &= R(b_1, a_1) \otimes \dots \otimes R(b_n, a_n) \\ &\leq R(\sigma(b_1, \dots, b_n), \sigma(a_1, \dots, a_n)) \\ &= R^{\circ}(\sigma(a_1, \dots, a_n), \sigma(b_1, \dots, b_n)) \end{aligned}$$

Next, we must confirm identities are preserved.

$$\begin{aligned} 1_A^{\circ}(a_1, a_2) &= 1_A(a_2, a_1) \\ &= \bigvee \{k \mid a_2 = a_1\} \\ &= 1_A(a_1, a_2) \end{aligned}$$

We confirm also functoriality with respect to composition

$$\begin{aligned} (R \circ S)^{\circ}(a, c) &= (R \circ S)(c, a) \\ &= \bigvee_b S(c, b) \otimes R(b, a) \\ &= \bigvee_b S^{\circ}(b, c) \otimes R^{\circ}(a, b) \\ &= \bigvee_b R^{\circ}(a, b) \otimes S^{\circ}(b, c) \\ &= (S^{\circ} \circ R^{\circ})(a, c) \end{aligned}$$

Finally, we must check that the converse distributes over tensors

$$\begin{aligned} (R^{\circ} \otimes S^{\circ})(b, b', a, a') &= R^{\circ}(b, a) \otimes S^{\circ}(b', a') \\ &= R(a, b) \otimes S(a', b') \\ &= (R \otimes S)(a, a', b, b') \\ &= (R \otimes S)^{\circ}(b, b', a, a') \end{aligned}$$

We moreover prove a fact used in the proof of proposition 3, that is, if f is an isomorphism in \mathcal{E} then

$$(f^{-1})_{\circ} = (f_{\circ})^{\circ}$$

This is a simple check:

$$\begin{aligned} (f^{-1})_{\circ}(b, a) &= \\ &= \bigvee \{k \mid f^{-1}b = a\} \\ &= \bigvee \{k \mid f(a) = b\} \\ &= f_{\circ}(a, b) \\ &= (f_{\circ})^{\circ}(b, a) \end{aligned} \quad \square$$

Theorem III.7. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \bigvee) an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a hypergraph category. The comonoid structure is given by the graphs of the canonical comonoids described in proposition 6, and the monoid structure is given by their converses.

Proof. For every object A of \mathcal{E} , call ϵ_A, δ_A the comultiplication and counit of proposition 6, and η_A, μ_A their respective converses. The morphisms ϵ_A, δ_A have in the internal logic the explicit form

$$\epsilon_A(a, x) = k$$

$$\delta_A(a_1, (a_2, a_3)) = \bigvee \{k \mid a_1 = a_2 = a_3\}$$

Checking that η_A, μ_A form a monoid is straightforward, from the definition of converse. With respect to this monoid/comonoid pair, we first confirm the special axiom

$$\begin{aligned} (\mu_A \circ \delta_A)(a_1, a_2) &= \\ &= \bigvee_{(a, a')} \delta_A(a_1, (a, a')) \otimes \mu_A((a, a'), a_2) \\ &= \bigvee_{(a, a')} \left[\bigvee \{k \mid a_1 = a = a'\} \right] \otimes \left[\bigvee \{k \mid a = a' = a_2\} \right] \\ &= \bigvee_{(a, a')} \bigvee \{k \mid a_1 = a = a' = a_2\} \\ &= \bigvee \{k \mid a_1 = a_2\} \\ &= 1_A(a_1, a_2) \end{aligned}$$

Finally, we check the Frobenius axiom, omitting some stages where the expressions get very long

$$\begin{aligned} ((1_A \otimes \delta_A) \circ (\mu_A \otimes 1_A))(a_1, a_2, a_3, a_4) &= \\ &= \bigvee_{x, y, z} 1_A(a_1, x) \otimes \delta_A(a_2, (y, z)) \otimes \mu_A((x, y), a_3) \otimes 1_A(z, a_4) \\ &= \bigvee_{x, y, z} \{k \mid x = y = z = a_1 = a_2 = a_3 = a_4\} \\ &= \bigvee \{k \mid a_1 = a_2 = a_3 = a_4\} \\ &= ((\delta_A \otimes 1_A) \circ (1_A \otimes \mu_A))(a_1, a_2, a_3, a_4) \end{aligned}$$

Lemma 1. *Let \mathcal{E} be a finitely complete category, and (Q, \otimes, k) an internal monoid. If*

$$h : (X_1, f_1, g_1, \chi_1) \rightarrow (X_2, f_2, g_2, \chi_2)$$

is a Q -span morphism with an inverse in \mathcal{E} , then it is an isomorphism.

Proof. We aim to show that h^{-1} is the required inverse as a Q -span morphism. We calculate

$$\begin{aligned} f_1 \circ h^{-1} &= f_2 \circ h \circ h^{-1} = f_2 \\ g_1 \circ h^{-1} &= g_2 \circ h \circ h^{-1} = g_2 \\ \chi_1 \circ h^{-1} &= \chi_2 \circ h \circ h^{-1} = \chi_2 \end{aligned}$$

Lemma 2. *Let \mathcal{E} be a finitely complete category and (Q, \otimes, k) an internal monoid. $\mathbf{Span}(Q)$ is a category.*

Proof. We first confirm that composition is independent of representatives. Consider span isomorphisms

$$\begin{aligned} \varphi : (X_1, f_1, g_1, \chi_1) &\rightarrow (X_2, f_2, g_2, \chi_2) \\ \psi : (Y_1, h_1, k_1, \xi_1) &\rightarrow (Y_2, h_2, k_2, \xi_2) \end{aligned}$$

We must show an isomorphism between the Q -spans

$$\begin{aligned} (X_1 \times_B Y_1, f_1 \circ p_1, k_1 \circ p_2, \otimes \circ (\chi_1 \times \xi_1) \circ \langle p_1, p_2 \rangle) \\ (X_2 \times_B Y_2, f_2 \circ p'_1, k_2 \circ p'_2, \otimes \circ (\chi_2 \times \xi_2) \circ \langle p'_1, p'_2 \rangle) \end{aligned}$$

We calculate, using properties of pullbacks

$$f_2 \circ p'_1 \circ \varphi \times_B \psi = f_2 \circ \varphi \circ p_1 = f_1 \circ p_1$$

$$g_2 \circ p'_2 \circ \varphi \times_B \psi = g_2 \circ \psi \circ p_2 = g_1 \circ p_2$$

Also

$$\begin{aligned} \otimes \circ (\chi_2 \times \xi_2) \circ \langle p'_1, p'_2 \rangle \circ \varphi \times_B \psi &= \\ &= \otimes \circ (\chi_2 \times \xi_2) \circ (\varphi \times \psi) \circ \langle p_1, p_2 \rangle \\ &= \otimes \circ (\chi_2 \circ \varphi, \xi_2 \circ \psi) \circ \langle p_1, p_2 \rangle \\ &= \otimes \circ (\chi_1 \times \psi_1) \circ \langle p_1, p_2 \rangle \end{aligned}$$

and as $\varphi \times_B \psi$ is also an isomorphism we can use lemma 1 to complete this part of the proof.

Next we confirm the left identity axiom. Firstly we note

$$\begin{aligned} (B, 1_B, 1_B, \chi_k) \circ (X, f, g, \chi) &= \\ &= (X \times_B B, f \circ p_1, p_2, \otimes \circ (\chi \times \chi_k) \circ \langle p_1, p_2 \rangle) \end{aligned}$$

We claim p_1 is a Q -span morphism to the span (X, f, g, χ) . The conditions for this being a span morphism are

$$f \circ p_1 = f \circ p_1 \quad \text{and} \quad g \circ p_1 = p_2$$

and the second condition is obvious from the pullback square. Finally, we must confirm this also commutes with the characteristic functions.

$$\begin{aligned} \otimes \circ (\chi \times \chi_k) \circ \langle p_1, p_2 \rangle &= p_1 \circ (\chi \times !) \circ \langle p_1, p_2 \rangle \\ &= p_1 \circ \langle \chi \circ p_1, ! \circ p_2 \rangle \\ &= \chi \circ p_1 \end{aligned}$$

□ We must prove that p_1 is an isomorphism, the required inverse being given by the universal property of pullbacks as

$$\langle 1, g \rangle$$

Checking that this is an isomorphism follows from the universal property of pullbacks. We can then use lemma 1 to complete this part of the proof. The right identity axiom follows similarly.

We must then confirm associativity. We consider the composites

$$\begin{aligned} L &= ((Z, l, m, \zeta) \circ (Y, h, k, \xi)) \circ (X, f, g, \chi) \\ R &= (Z, l, m, \zeta) \circ ((Y, h, k, \xi) \circ (X, f, g, \chi)) \end{aligned}$$

Via the usual proof for categories of ordinary spans

$$\iota := \langle p_1 \circ p_1, \langle p_2 \circ p_1, p_2 \rangle \rangle : (X \times_B Y) \times_C Z \rightarrow X \times_B (Y \times_C Z)$$

is an isomorphism of spans. It remains to show that this commutes with the characteristic morphisms. This is a horrible exercise in tracking various canonical morphisms, and is most easily handled using the graphical calculus for a cartesian monoidal category. Details are omitted to avoid a long type-setting exercise for the diagrams. □

Lemma 3. *Let \mathcal{E} be a finitely complete category and (Q, \otimes, k) an internal monoid. There is an identity on objects contravariant involution (dagger functor) given by*

$$\begin{aligned} (-)^\circ : \mathbf{Span}(Q)^{op} &\rightarrow \mathbf{Span}(Q) \\ (X, f, g, \chi) &\mapsto (X, g, f, \chi) \end{aligned}$$

Proof. This involution clearly preserves identities. We aim to show

$$(X, f, g, \chi)^\circ \circ (Y, h, k, \xi)^\circ = ((Y, h, k, \xi) \circ (X, f, g, \chi))^\circ$$

These two spans are given by

$$\begin{aligned} (X, f, g, \chi)^\circ \circ (Y, h, k, \xi)^\circ &= \\ &= (Y \times_B X, k \circ p_1, f \circ p_2, \otimes \circ \langle \xi \circ p_1, \chi \circ p_2 \rangle) \\ ((Y, h, k, \xi) \circ (X, f, g, \chi))^\circ &= \\ &= (X \times_B Y, k \circ p_2, f \circ p_1, \otimes \circ \langle \chi \circ p_1, \xi \circ p_2 \rangle) \end{aligned}$$

The morphisms

$$\begin{aligned} \langle p_2, p_1 \rangle : X \times_B Y &\rightarrow Y \times_B X \\ \langle p_2, p_1 \rangle : Y \times_B X &\rightarrow X \times_B Y \end{aligned}$$

witness an isomorphism in \mathcal{E} . We first confirm that this gives a span isomorphism. This follows trivially from elementary properties of pullbacks. Finally, we must prove that this commutes with characteristic morphisms. This makes essential use of commutativity of \otimes

$$\begin{aligned} \otimes \circ \langle \xi \circ p_1, \chi \circ p_2 \rangle \circ \langle p_2, p_1 \rangle &= \otimes \circ \langle \xi \circ p_2, \chi \circ p_1 \rangle \\ &= \otimes \circ \langle \chi \circ p_1, \xi \circ p_2 \rangle \quad \square \end{aligned}$$

Lemma 4. *Let \mathcal{E} be a finitely complete category, and (Q, \otimes, k) an internal monoid. There is an identity on objects covariant graph functor*

$$\begin{aligned} (-)_\circ : \mathcal{E} &\rightarrow \mathbf{Span}(Q) \\ f : A \rightarrow B &\mapsto (A, 1_A, f, \chi_k) \end{aligned}$$

There is also an identity on objects contravariant cograph functor

$$\begin{aligned} {}^\circ(-) : \mathcal{E}^{op} &\rightarrow \mathbf{Span}(Q) \\ f : A \rightarrow B &\mapsto (B, f, 1_B, \chi_k) \end{aligned}$$

If Q is a commutative monoid then

$${}^\circ(-) = (-)^\circ \circ (-)_\circ$$

Proof. This construction is well known for ordinary spans. For Q -spans, we must confirm the interaction with characteristic morphisms behaves appropriately. Firstly we note that

$$(1_A)_\circ = (A, 1_A, 1_A, \chi_k)$$

and so identities are preserved. For composition, we have an ordinary span isomorphism

$$\langle 1_A, f \rangle : A \rightarrow A \times_B B$$

We must confirm this commutes with characteristic morphisms

$$\begin{aligned} \otimes \circ (\chi_k \times \chi_k) \circ \langle p_1, p_2 \rangle \circ \langle 1_A, f \rangle &= \otimes \circ \langle \chi_k, \chi_k \circ f \rangle \\ &= \otimes \circ \langle \chi_k, \chi_k \rangle \\ &= \chi_k \end{aligned}$$

The proof for the cograph construction is similar. In the case of commutative Q , the relationship to the converse functor is immediate from the definitions. \square

Lemma 5. *Let \mathcal{E} be a finitely complete category and (Q, \otimes, k) an internal commutative monoid. There is a bifunctor*

$$\begin{aligned} \otimes : \mathbf{Span}(Q) \times \mathbf{Span}(Q) &\rightarrow \mathbf{Span}(Q) \\ A \otimes B &= A \times B \end{aligned}$$

$$\begin{aligned} (X_1, f_1, g_1, \chi_1) \otimes (X_2, f_2, g_2, \chi_2) &= \\ &= (X_1 \times X_2, f_1 \times f_2, g_1 \times g_2, \otimes \circ (\chi_1 \times \chi_2)) \end{aligned}$$

Furthermore, this bifunctor commutes with graphs in the sense that the following diagram commutes

$$\begin{array}{ccc} \mathbf{Span}(Q) \times \mathbf{Span}(Q) & \xrightarrow{\otimes} & \mathbf{Span}(Q) \\ \uparrow (-)_\circ \times (-)_\circ & & \uparrow (-)_\circ \\ \mathcal{E} \times \mathcal{E} & \xrightarrow{\times} & \mathcal{E} \end{array}$$

Proof. We first show that this respects equivalence classes of spans. Assume we have span isomorphisms

$$\begin{aligned} \varphi : (X_1, f_1, g_1, \chi_1) &\rightarrow (X'_1, f'_1, g'_1, \chi'_1) \\ \psi : (X_2, f_2, g_2, \chi_2) &\rightarrow (X'_2, f'_2, g'_2, \chi'_2) \end{aligned}$$

The product $\varphi \times \psi$ gives an isomorphism of ordinary spans

$$\begin{aligned} \varphi \times \psi : (X_1, f_1, g_1, \chi_1) \otimes (X_2, f_2, g_2, \chi_2) &\rightarrow \\ &\rightarrow (X'_1, f'_1, g'_1, \chi'_1) \otimes (X'_2, f'_2, g'_2, \chi'_2) \end{aligned}$$

It then remains to check this commutes with characteristic morphisms. We calculate

$$\begin{aligned} \otimes \circ (\chi'_1 \times \chi'_2) \circ (\varphi \times \psi) &= \otimes \circ [(\chi'_1 \circ \varphi) \times (\chi'_2 \circ \psi)] \\ &= \otimes \circ (\chi_1 \times \chi_2) \end{aligned}$$

That this is bifunctorial as an operation on the underlying spans is well known. It remains to check the behaviour with respect to characteristic morphisms. For identity Q -spans, the resulting characteristic function is

$$\otimes \circ (\chi_k \times \chi_k) = \chi_k$$

For composition, we note there is an isomorphism of spans

$$\begin{aligned} \langle \langle p_1 \circ p_1, p_1 \circ p_2 \rangle, \langle p_2 \circ p_1, p_2 \circ p_2 \rangle \rangle : \\ : (X \times_B Y) \times (X' \times_{B'} Y') \rightarrow (X \times X') \times_{B \times B'} (Y \times Y') \end{aligned}$$

We must show this commutes with the corresponding characteristic morphisms. The following unpleasant calculation establishes the required equality

$$\begin{aligned} \otimes \circ (\otimes \times \otimes) \circ [((\chi \times \xi) \circ \langle p_1, p_2 \rangle) \times ((\chi' \times \xi') \circ \langle p_1, p_2 \rangle)] &= \\ &= \otimes \circ (\otimes \times \otimes) \circ [(\chi \times \xi) \times (\chi' \times \xi')] \circ \\ &\quad \circ \langle \langle p_1 p_1, p_2 p_1 \rangle, \langle p_1 p_2, p_2 p_2 \rangle \rangle \\ &= \otimes \circ (\otimes \times \otimes) \circ [(\chi \times \chi') \times (\xi \times \xi')] \circ \\ &\quad \circ \langle \langle p_1 p_1, p_1 p_2 \rangle, \langle p_2 p_1, p_2 p_2 \rangle \rangle \circ \\ &\quad \circ \langle \langle p_1 p_1, p_2 p_1 \rangle, \langle p_1 p_2, p_2 p_2 \rangle \rangle \\ &= \otimes \circ (\otimes \times \otimes) \circ [(\chi \times \chi') \times (\xi \times \xi')] \circ \end{aligned}$$

$$\circ \langle \langle p_1 p_1, p_1 p_2 \rangle, \langle p_2 p_1, p_2 p_2 \rangle \rangle$$

Finally, we must confirm that this bifunctor commutes with graphs. On objects this is obvious, as all the functors involved act as the identity on objects. On morphisms we reason

$$\begin{aligned} (f_1 \times f_2)_\circ &= (A_1 \times A_2, 1_{A_1 \times A_2}, f_1 \times f_2, k) \\ &= (A_1, 1_{A_1}, f, k) \otimes (A_2, 1_{A_2}, f, k) \\ &= (f_1)_\circ \otimes (f_2)_\circ \end{aligned}$$

□

Lemma 6. *Let \mathcal{E} be a finitely complete category and (Q, \otimes, k) an internal monoid. The following equation holds in $\mathbf{Span}(Q)$*

$$k_\circ \circ (X, f, g, \chi) \circ \circ h = (X, h \circ f, k \circ g, \chi)$$

Proof. We firstly consider the case of post composition with the graph of a morphism in the underlying category

$$k_\circ \circ (X, f, g, \chi)$$

This composite is given by the pullback span

$$(X \times_B B, p_1 \circ f, p_2 \circ k, \otimes \circ (\chi \times \chi_k) \circ \langle p_1, p_2 \rangle)$$

We note that $p_1 \circ \langle 1_X, g \rangle = 1_X$ and

$$\langle 1_X, g \rangle \circ p_1 = \langle p_1, g \circ p_1 \rangle = \langle p_1, p_2 \rangle = 1_{X \times_B B}$$

and so p_1 and $\langle 1_X, g \rangle$ witness an isomorphism in \mathcal{E} . We next confirm they give a span isomorphism. One of the conditions for p_1 to be a span morphism is trivial, for the other

$$k \circ g \circ p_1 = k \circ 1 \circ p_2 = k \circ p_2$$

Finally, we must confirm that this commutes with the characteristic morphisms

$$\otimes \circ (\chi \times \chi_k) \circ \langle p_1, p_2 \rangle = \chi \circ p_1 \circ \langle p_1, p_2 \rangle = \chi \circ p_1$$

Now we note that

$$\begin{aligned} (X, f, g, \chi) \circ \circ h &= (X, f, g, \chi) \circ h_\circ^\circ \\ &= (h_\circ \circ (X, f, g, \chi))^\circ \\ &= (h_\circ \circ (X, g, f, \chi))^\circ \\ &= (X, g, h \circ f, \chi)^\circ \\ &= (X, h \circ f, g, \chi) \end{aligned}$$

Combining these two observations then completes the proof. □

Theorem IV.3. *Let \mathcal{E} be a finitely complete category, and (Q, \otimes, k) an internal commutative monoid. The category $\mathbf{Span}(Q)$ is a hypergraph category.*

Proof. We first confirm that $\mathbf{Span}(Q)$ carries a monoidal structure. We take as our monoidal unit the terminal object in \mathcal{E} and the tensor to be the bifunctor of proposition 5. Next, we note that the underlying category is a symmetric monoidal category with respect to binary products. The graph construction is surjective on objects, and commutes with the tensor, therefore the graphs of the coherence morphisms in \mathcal{E} lift to $\mathbf{Span}(Q)$. We must confirm that each of these remains natural in $\mathbf{Span}(Q)$.

Applying proposition 6, it is sufficient to show the following are span isomorphisms

$$\begin{aligned} \lambda_X : (1 \times X, \lambda_A \circ (1_1 \times f), \lambda_B \circ (1_1 \times g), \\ , \otimes \circ (\chi_k \times \chi)) \rightarrow (X, f, g, \chi) \end{aligned}$$

$$\begin{aligned} \rho_X : (X \times 1, \rho_A \circ (f \times 1_1), \rho_B \circ (g \times 1_1), \\ , \otimes \circ (\chi \times \chi_k)) \rightarrow (X, f, g, \chi) \end{aligned}$$

$$\begin{aligned} \alpha_{X,Y,Z} : ((X_1 \times X_2) \times X_3, \alpha_{A_1,A_2,A_3} \circ \\ \circ ((f_1 \times f_2) \times f_3), \alpha_{B_1,B_2,B_3} \circ ((g_1 \times g_2) \times g_3), \dots) \rightarrow \\ \rightarrow (X_1 \times (X_2 \times X_3)) \end{aligned}$$

$$\begin{aligned} \sigma_{X,Y} : (X_1 \times X_2, \sigma_{A_1,A_2} \circ f_1 \times f_2, \sigma_{B_1,B_2} \circ g_1 \times g_2, \\ , \otimes \circ (\chi_1 \times \chi_2)) \rightarrow \\ \rightarrow (X_2 \times X_1, f_2 \times f_1, g_2 \times g_1, \otimes \circ (\chi_2 \times \chi_1)) \end{aligned}$$

And this is now just a straightforward (but very unpleasant) check. □

Lemma 7. *Let \mathcal{E} be a topos, (Q, \otimes, k, \leq) an internal partially ordered commutative monoid, and (Σ, E) an algebraic variety. If (X_1, f_1, g_1, χ_1) is an algebraic Q -span, and (X_2, f_2, g_2, χ_2) is an isomorphic Q -span, then it is also an algebraic span.*

Proof. For the assumptions in the question, with ι denoting the assumed isomorphism and $\sigma \in \Sigma$, if

$$f_2(x_1) = a_1 \wedge g_2(x_1) = b_1 \wedge \dots \wedge f_2(x_n) = a_n \wedge g_2(x_n) = b_n$$

then using our span isomorphism.

$$\begin{aligned} f_1(\iota^{-1}(x_1)) &= a_1 \wedge g_1(\iota^{-1}(x_1)) = b_1 \wedge \dots \\ &\dots \wedge f_1(\iota^{-1}(x_n)) = a_n \wedge g_1(\iota^{-1}(x_n)) = b_n \end{aligned}$$

By assumption that the first span is algebraic, there exists x such that

$$\begin{aligned} f_1(x) &= \sigma(a_1, \dots, a_n) \wedge g_1(x) = \sigma(b_1, \dots, b_n) \wedge \\ &\wedge \chi_1(\iota^{-1}(x_1)) \otimes \dots \otimes \chi_1(\iota^{-1}(x_n)) \leq \chi_1(x) \end{aligned}$$

Therefore, using our span isomorphism again

$$\begin{aligned} f_2(\iota(x)) &= \sigma(a_1, \dots, a_n) \wedge g_2(\iota(x)) = \sigma(b_1, \dots, b_n) \wedge \\ &\wedge \chi_2(x_1) \otimes \dots \otimes \chi_2(x_n) \leq \chi_2(\iota(x)) \end{aligned}$$

□

Lemma 8. *Let \mathcal{E} be a topos, (Q, \otimes, k, \leq) an internal partially ordered commutative monoid, and (Σ, E) an algebraic variety. $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a category.*

Proof. Throughout, we will perform checks for an arbitrary $\sigma \in \Sigma$. Firstly, we must confirm that the identity morphisms are algebraic. The required condition is immediate, as A is closed under the algebraic operations, and the characteristic morphism is constant in the internal language.

Secondly, we must confirm that algebraic Q -spans are closed under composition. Assume

$$\begin{aligned} f(p_1(z_1)) &= a_1 \wedge k(p_2(z_1)) = c_1 \wedge \dots \\ &\dots \wedge f(p_1(z_n)) = a_n \wedge k(p_2(z_n)) = c_n \end{aligned}$$

then there exist x_1, \dots, x_n and y_1, \dots, y_n such that

$$f(x_1) = a_1 \wedge g(x_1) = h(y_1) \wedge k(y_1) = c_1 \wedge \dots \\ \dots \wedge f(x_n) = a_n \wedge g(x_n) = h(y_n) \wedge k(y_n) = c_n$$

therefore as the component spans are algebraic, there exist x and y such that

$$f(x) = \sigma(a_1, \dots, a_n) \wedge g(x) = h(y) \wedge k(y) = \sigma(c_1, \dots, c_n)$$

and both

$$\chi(x_1) \otimes \dots \otimes \chi(x_n) \leq \chi(x), \quad \xi(y_1) \otimes \dots \otimes \xi(y_n) \leq \xi(y)$$

Therefore we have (x, y) in the apex of the composite span, with truth value $\chi(x) \otimes \xi(y)$. By monotonicity of the tensor

$$\chi(x_1) \otimes \dots \otimes \chi(x_n) \otimes \xi(y_1) \otimes \dots \otimes \xi(y_n) \leq \chi(x) \otimes \xi(y)$$

Finally, as the tensor is commutative

$$\chi(x_1) \otimes \xi(y_1) \otimes \dots \otimes \chi(x_n) \otimes \xi(y_n) \leq \chi(x) \otimes \xi(y)$$

That the composition is associative and the identities satisfy the required axioms follows immediately from the same properties for the underlying Q -spans as established in lemma 2. \square

Proposition 5. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a symmetric monoidal category. The symmetric monoidal structure is inherited from the finite product structure in \mathcal{E} .*

Proof. Throughout, we will perform checks for an arbitrary $\sigma \in \Sigma$. The proof is exactly as in lemma 5, we just need to prove that the functors defined in lemmas 3, 4, 5 respect the algebraic condition.

With regard to lemma 3, we just observe that the condition for a Q -span to be algebraic is symmetrical in its domain and codomain, and therefore preserved.

With regard to lemma 4, as the characteristic morphism is constant, we must simply confirm the existence of witnesses relating composite terms. If

$$f(a_1) = b_1 \wedge \dots \wedge f(a_n) = b_n$$

then as f is a homomorphism

$$f(\sigma(a_1, \dots, a_n)) = \sigma(f(a_1), \dots, f(a_n)) = \sigma(b_1, \dots, b_n)$$

With regard to lemma 5, for algebraic Q -spans (X, f, g, χ) and (X', f', g', χ') , if

$$(f \times f')(x_1, x'_1) = (a_1, a'_1) \wedge \dots \wedge (f \times f')(x_n, x'_n) = (a_n, a'_n)$$

and

$$(g \times g')(x_1, x'_1) = (b_1, b'_1) \wedge \dots \wedge (g \times g')(x_n, x'_n) = (b_n, b'_n)$$

Then

$$f(x_1) = a_1 \wedge f'(x'_1) = a'_1 \wedge \dots \wedge f(x_n) = a_n \wedge f'(x'_n) = a'_n$$

and

$$g(x_1) = b_1 \wedge g'(x'_1) = b'_1 \wedge \dots \wedge g(x_n) = b_n \wedge g'(x'_n) = b'_n$$

As the spans are algebraic, there exist x and x' such that

$$\chi(x_1) \otimes \dots \otimes \chi(x_n) \leq \chi(x), \quad \chi'(x'_1) \otimes \dots \otimes \chi'(x'_n) \leq \chi'(x')$$

By monotonicity and commutativity of the tensor, we then have

$$\chi(x_1) \otimes \chi'(x'_1) \otimes \dots \otimes \chi(x_n) \otimes \chi'(x'_n) \leq \chi(x) \otimes \chi'(x')$$

As we said, functoriality, and that the required diagram commutes then follows from the proof of theorem IV.3 as tensors coincide on the underlying Q -spans. \square

Proposition 6. *[Converse] Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. There is an identity on objects strict symmetric monoidal functor*

$$(-)^\circ : \mathbf{Span}_{(\Sigma, E)}(Q)^{op} \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q)$$

given by reversing the legs of the underlying span:

$$(X, f, g, \chi)^\circ = (X, g, f, \chi)$$

Proof. Converse is defined as in lemma 3, and the proof that this is algebraic is in proposition 5. That converse commutes with the tensor is trivial. \square

Proposition 7. *[Graph] Let \mathcal{E} be a topos, and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. There is an identity on objects strict symmetric monoidal functor*

$$(-)_\circ : \mathbf{Alg}(\Sigma, E) \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q)$$

with the action on morphism $f : A \rightarrow B$ given by

$$f_\circ = (A, 1, f, \chi_k)$$

Proof. Graph is defined as in lemma 4, and the proof that this is algebraic is in proposition 5. The proof that graphs commute with the tensor is as in lemma 5. \square

Theorem IV.8. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a hypergraph category. The cocommutative comonoid structure is given by the graphs of the canonical comonoids described in proposition 6, and the monoid structure is given by their converses.*

Proof. As in theorem IV.3, since all the tools used there preserve the algebraic structure. \square

Theorem V.2. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \vee) an internal commutative quantale. The category $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is a partially ordered symmetric monoidal category.*

Proof. First of all we have to prove that our partial order is well defined. It is clearly reflexive. For transitivity, if

$$R \subseteq R' \quad \text{and} \quad R' \subseteq R''$$

then both

$$\vdash R(a, b) \leq R'(a, b) \quad \text{and} \quad \vdash R'(a, b) \leq R''(a, b)$$

Therefore internally

$$\vdash R(a, b) \leq R'(a, b) \wedge R'(a, b) \leq R''(a, b)$$

and so by transitivity of the order on the quantale

$$\vdash R(a, b) \leq R''(a, b)$$

Finally, if $R \subseteq R'$ and $R' \subseteq R$, and so internally

$$\vdash R(a, b) \leq R'(a, b) \wedge R'(a, b) \leq R(a, b)$$

by antisymmetry of the order on the internal quantale

$$\vdash R(a, b) = R'(a, b)$$

and so

$$\vdash \forall a, b. R(a, b) = R'(a, b)$$

Meaning externally $R = R'$.

Next, we must confirm that composition is monotone in both components. As the proofs are symmetrical, we only consider precomposition explicitly.

Assume $R \subseteq R'$. We consider post-composing each of these relations with the relation S . Remembering that the quantale product preserves order, we calculate

$$\begin{aligned} (S \circ R)(a, c) &= \bigvee_b R(a, b) \otimes S(b, c) \\ &\leq R'(a, b) \otimes S(b, c) \\ &= \bigvee_b R'(a, b) \otimes S(b, c) \\ &= (S \circ R')(a, c) \end{aligned}$$

Finally, we must confirm that the tensor on $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is monotone in both arguments. Assume again $R \subseteq R'$. We calculate

$$\begin{aligned} (R \otimes S)(a, b, c, d) &= R(a, b) \otimes S(c, d) \\ &\leq R'(a, b) \otimes S(c, d) \\ &= (R' \otimes S)(a, b, c, d) \end{aligned} \quad \square$$

Theorem V.4. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. The category $\mathbf{Span}_{(\Sigma, E)}(Q)$ is a preordered symmetric monoidal category.*

Proof. Firstly, we must confirm that this ordering is independent of choices of representatives for equivalence classes of spans.

Assume $(X_1, f_1, g_1, \chi_1) \subseteq (Y, h_1, k_1, \xi_1)$, and span isomorphisms

$$\begin{aligned} i &: (X_1, f_1, g_1, \chi_1) \rightarrow (X_2, f_2, g_2, \chi_2) \\ j &: (Y_1, h_1, k_1, \xi_1) \rightarrow (Y_2, h_2, k_2, \xi_2) \end{aligned}$$

Let

$$m : (X_1, f_1, g_1, \chi_1) \rightarrow (Y, h_1, k_1, \xi_1)$$

be the span morphism that is monic in \mathcal{E} required by the assumed order structure. There is then a span morphism

$$j \circ m \circ i^{-1} : (X_2, f_2, g_2, \chi_2) \rightarrow (Y_2, h_2, k_2, \xi_2)$$

which is monic in \mathcal{E} as monomorphisms are closed under composition. We then have

$$\chi_2(x) = \chi_1(i^{-1}(x)) = \xi_1(m \circ i^{-1}(x)) = \xi_2(j \circ m \circ i^{-1}(x))$$

The relation \subseteq is clearly reflexive via the identity Q -span morphism.

For transitivity, assume

$$(X, f, g, \chi) \subseteq (Y, h, k, \xi) \subseteq (Z, m, n, \zeta)$$

Denote the required monomorphisms

$$\begin{aligned} r &: (X, f, g, \chi) \rightarrow (Y, h, k, \xi) \\ s &: (Y, h, k, \xi) \rightarrow (Z, m, n, \zeta) \end{aligned}$$

There is then an obvious span morphism $s \circ r$ that is a monomorphism in \mathcal{E} . We then have

$$\chi(x) \leq \xi(r(x))$$

and so

$$\xi(r(x)) \leq \zeta(s \circ r(x))$$

By transitivity of the quantale ordering

$$\chi(x) \leq \zeta(s \circ r(x))$$

Then, we must confirm that composition is monotone in both components. As the proofs are symmetrical, we only consider precomposition explicitly.

Assume $(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$ as witnessed by \mathcal{E} monomorphism $m : X_1 \rightarrow X_2$. We consider post-composing each of these spans with the span (Y, h, k, ξ) . There is then a Q -span morphism

$$\begin{aligned} m \times_B 1 : \{(x_1, y) \mid f_1(x_1) = h(y)\} &\rightarrow \\ &\rightarrow \{(x_2, y) \mid f_2(x_2) = h(y)\} \end{aligned}$$

and the underlying morphism is a monomorphism in \mathcal{E} by standard properties of pullbacks and monomorphisms. By monotonicity of the tensor

$$\begin{aligned} (\otimes \circ (\chi_1 \times \xi) \circ \langle p_1, p_2 \rangle)(x, y) &= \xi_1(x) \otimes \xi(y) \\ &\leq \xi_2(m(x)) \otimes \xi(y) \\ &= (\otimes \circ (\chi_2 \times \xi) \circ \langle p_1, p_2 \rangle)(m(x), y) \\ &= (\otimes \circ (\chi_2 \times \xi) \circ \langle p_1, p_2 \rangle \circ (m \times_B 1))(x, y) \end{aligned}$$

Finally, we must confirm that the tensor bifunctor on $\mathbf{Span}_{(\Sigma, E)}(Q)$ is monotone in both arguments. Assume

$$(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$$

witnessed by \mathcal{E} monomorphism $m : X_1 \rightarrow X_2$. There is then a span morphism

$$\begin{aligned} m \times 1 : (X_1, f_1, g_1, \chi_1) \otimes (Y, h, k, \xi) &\rightarrow \\ &\rightarrow (X_2, f_2, g_2, \chi_2) \otimes (Y, h, k, \xi) \end{aligned}$$

and this a \mathcal{E} monomorphism by standard theory of products and monomorphisms. We then calculate

$$\otimes \circ (\chi_1 \times \xi)(x, y) = \chi_1(x) \otimes \xi(y)$$

$$\begin{aligned} &\leq \chi_2(m(x)) \otimes \xi(y) \\ &= (\otimes \circ (\chi_2 \times \xi) \circ (m \times 1))(x, y) \quad \square \end{aligned}$$

Theorem VI.1. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. There is an identity on objects, strict symmetric monoidal **Preord**-functor

$$V : \mathbf{Span}_{(\Sigma, E)}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q)$$

Proof. We define the action on morphisms on a chosen representative span as follows

$$V(X, p_1, p_2, \chi)(a, b) = \bigvee_x \{ \chi(x) \mid p_1(x) = a \wedge p_2(x) = b \}$$

It is easy to check that this definition is independent of our choice of representatives. We moreover check preservation of identities:

$$\begin{aligned} V(A, 1_A, 1_A, \chi_k)(a_1, a_2) &= \bigvee \{ \chi_k(a) \mid 1_A(a) = a_1 \wedge 1_A(a) = a_2 \} \\ &= \bigvee \{ k \mid a = a_1 \wedge a = a_2 \} \\ &= \bigvee \{ k \mid a_1 = a_2 \} \\ &= 1_A(a_1, a_2) \end{aligned}$$

That the functor commutes with the tensor is clear from the definition. That the coherence morphisms for the monoidal structures are preserved on the nose is clear as they were constructed using the graph constructions, and V preserves graphs. To see that V preserves preorders, we assume

$$(X_1, f_1, g_1, \chi_1) \subseteq (X_2, f_2, g_2, \chi_2)$$

witnessed by a monomorphism $m : X_1 \rightarrow X_2$. We then calculate

$$\begin{aligned} &\bigvee \{ \chi_1(x) \mid f_1(x) = a \wedge g_1(x) = b \} \leq \\ &\leq \bigvee \{ \chi_2(m(x)) \mid f_2(m(x)) = a \wedge g_2(m(x)) = b \} \\ &\leq \bigvee \{ \chi_2(x') \mid f_2(x') = a \wedge g_2(x') = b \} \quad \square \end{aligned}$$

Theorem VII.1. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and $h : Q_1 \rightarrow Q_2$ a morphism of internal commutative quantales. There is an identity on objects, strict symmetric monoidal **Pos**-functor

$$h^* : \mathbf{Rel}_{(\Sigma, E)}(Q_1) \rightarrow \mathbf{Rel}_{(\Sigma, E)}(Q_2)$$

The assignment $h \mapsto h^*$ is functorial.

Proof. h^* acts by postcomposing a relation $R : A \times B \rightarrow Q_1$ with h :

$$h^*(R) : A \times B \xrightarrow{R} Q_1 \xrightarrow{h} Q_2$$

Preservation of identities, compositions and tensors follows from the fact that h preserves quantale identities, products and joins. In particular, the preservation of quantale joins makes h also order-preserving, proving that h^* is a **Pos**-functor. The functoriality of assignments is trivial: Postcomposing with the identity function on a quantale gives back the same category we started from, and the composition of quantale homomorphisms is a quantale homomorphism. \square

Theorem VII.3. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and $h : Q_1 \rightarrow Q_2$ a morphism of internal partially ordered commutative monoids. There is an identity on objects, strict symmetric monoidal **Preord**-functor

$$h^* : \mathbf{Span}_{(\Sigma, E)}(Q_1) \rightarrow \mathbf{Span}_{(\Sigma, E)}(Q_2)$$

The assignment $h \mapsto h^*$ is functorial.

Proof. The h^* functor is again defined by postcomposition, as in the relational case. Proof of functoriality and preservation of tensor and preorder is almost identical to the relational case. \square

Proposition 8. Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. For linear (affine, relevant, cartesian) algebraic Q -relation $R : A \rightarrow B$ the axiom

$$R(a_1, b_1) \otimes \dots \otimes R(a_n, b_n) \leq R(\tau(a_1, \dots, a_n), \tau(b_1, \dots, b_n))$$

holds for every linear (affine, relevant, cartesian) n -ary derived operation τ .

Proof. Suppose R is a linear relation. we proceed by induction. By definition, if τ is any n -ary operation, then it is

$$\bigotimes_{k=1}^n R(a_k, b_k) \leq R(\tau(a_1, \dots, a_n), \tau(b_1, \dots, b_n))$$

(in this proof the big tensor symbol is just a shorthand for the quantale product over a finite number of components). Now let τ_1, \dots, τ_n be operations of arities n, k_1, \dots, k_n , respectively. Being τ an operation it is

$$\begin{aligned} &\bigotimes_{i=1}^n R(\tau_i(a_1^i, \dots, a_{k_i}^i), \tau_i(b_1^i, \dots, b_{k_i}^i)) \leq \\ &\leq R(\tau(\tau_1(a_1^1, \dots, a_{k_1}^1), \dots, \tau_n(a_1^n, \dots, a_{k_n}^n)), \\ &\quad \tau(\tau_1(b_1^1, \dots, b_{k_1}^1), \dots, \tau_n(b_1^n, \dots, b_{k_n}^n))) \end{aligned}$$

and combining with the same condition on the τ_i one obtains

$$\begin{aligned} &\bigotimes_{i=1}^n \bigotimes_{z=1}^{k_i} R(a_z^i, b_z^i) \leq \\ &\leq R(\tau(\tau_1(a_1^1, \dots, a_{k_1}^1), \dots, \tau_n(a_1^n, \dots, a_{k_n}^n)), \\ &\quad \tau(\tau_1(b_1^1, \dots, b_{k_1}^1), \dots, \tau_n(b_1^n, \dots, b_{k_n}^n))) \end{aligned}$$

This concludes the proof since every linear term can be written as a concatenation of operations.

Affine terms are obtained as compositions of operations and projections. It is thus sufficient to prove that the condition holds for affine relations if τ is a projection. Then, we can treat any n -ary projection as a generic operation and proceed as in the previous case. But the condition

$$\bigotimes_{k=1}^n R(a_k, b_k) \leq R(\pi(a_1, \dots, a_n), \pi(b_1, \dots, b_n))$$

being the right hand side just $R(a_i, b_i)$, trivially holds when R is affine.

Relevant terms are obtained as compositions of operations and diagonals. Note that a diagonal δ is not a term when taken alone because it is not a morphism of the form $X^n \rightarrow X$ for some n . This means that if a term is built using diagonals, there is always at least one operation that is composed with the diagonal on the left. The proof is then very similar to the previous ones, with the additional step that if R is relevant, then the condition has to be proven to hold for every $\tau(x_1, \dots, x_i, \delta(x), x_{i+1}, \dots, x_n)$, where δ is the n -th diagonal and τ is any $(n+m)$ -ary operation.

For cartesian terms it is sufficient to put all these observation together and proceed in the same way. \square

Proposition 9. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} and (Q, \otimes, k, \vee) an internal commutative quantale. If Q is linear (affine, relevant, cartesian), every morphism in $\mathbf{Rel}_{(\Sigma, E)}(Q)$ is linear (affine, relevant, cartesian).*

Proof. We only prove the affine case explicitly, all the rest being similar. For arbitrary a_1, a_2, b_1, b_2 consider the product $R(a_1, b_1) \otimes R(a_2, b_2)$. Both $R(a_1, b_1)$ and $R(a_2, b_2)$ are elements of Q , hence, being Q affine, we can readily infer $R(a_1, b_1) \otimes R(a_2, b_2) \leq R(a_1, b_1)$. Being the variables arbitrarily chosen, we universally quantify on them obtaining the axiom of being affine for R as a valid formula in the ambient topos \mathcal{E} . \square

Theorem VIII.10. *Let \mathcal{E} be a topos and (Q, \otimes, k, \vee) an internal commutative quantale. Let $i : (\Sigma_1, E_1) \rightarrow (\Sigma_2, E_2)$ be a linear interpretation of signatures. There is a strict monoidal functor*

$$i^* : \mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) \rightarrow \mathbf{Rel}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)$$

The assignment $i \mapsto i^*$ extends to a contravariant functor.

Similar results hold for affine, relevant and cartesian interpretations and relations.

Proof. An object of $\mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q)$ is written as $\langle\langle A, \sigma_j \rangle\rangle$, where A is an object of \mathcal{E} and the σ_j are morphisms $A^n \rightarrow A$ in bijective correspondence with the operations in Σ_2 , agreeing with them on arities, and such that they satisfy the equations in E_2 (these equations are just commutative diagrams between the above mentioned morphisms). The linear (affine, relevant, cartesian) interpretation i maps every operation in $\sigma'_k \in \Sigma_1$ to a linear (affine, relevant cartesian) term $i(\sigma'_k)$ on Σ_2 , such that these terms satisfy the equations in E_1 . This means that $\langle\langle A, i(\sigma'_k) \rangle\rangle$ is an algebra of type (Σ_1, E_1) .

The functor i^* then acts as follows: It sends every $\langle\langle A, \sigma_j \rangle\rangle$ to $\langle\langle A, i(\sigma'_k) \rangle\rangle$, and it is identity on morphisms (the fact that morphisms of $\mathbf{Rel}_{(\Sigma_2, E_2)}^{\text{lin}}(Q)$ are also morphisms of $\mathbf{Rel}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)$ is a direct consequence of proposition 8). Functoriality then holds trivially being i^* identity on morphisms. \square

Proposition 13. *Let \mathcal{E} be a topos, (Σ, E) a variety in \mathcal{E} , and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. For (Σ, E) -algebras A and B , and linear (affine,*

relevant, cartesian) algebraic Q -span (X, f, g, χ) and n -ary linear (affine, relevant, cartesian) term τ if

$$f(x_1) = a_1 \wedge g(x_1) = b_1 \wedge \dots \wedge f(x_n) = a_n \wedge g(x_n) = b_n$$

then there exists x such that

$$f(x) = \tau(a_1, \dots, a_n) \wedge g(x) = \tau(b_1, \dots, b_n)$$

and

$$\chi(x_1) \otimes \dots \otimes \chi(x_n) \leq \chi(x)$$

Proof. As in the relational case, we proceed by induction. Let (X, f, g, χ) be a span. By definition, if τ is any n -ary operation, then the axiom (here the big wedge and the big tensor product are just a shorthand for a logical conjunction and a quantale product over a finite number of components, respectively)

$$\begin{aligned} \bigwedge_{i=1}^n (f(x_i) = a_i \wedge g(x_i) = b_i) &\implies \\ \exists x : f(x) = \tau(a_1, \dots, a_n), k(x) \wedge g(x) = \tau(b_1, \dots, b_n) \wedge \\ &\wedge \bigotimes_{i=1}^n \chi(x_i) \leq \chi(x) \end{aligned}$$

is already satisfied. Now let $\tau, \tau_1, \dots, \tau_n$ be operations of arities n, k_1, \dots, k_n , respectively. Being τ an operation it is

$$\begin{aligned} \bigwedge_{i=1}^n (f(x^i) = \tau_i(a_1^i, \dots, a_{k_i}^i) \wedge g(x^i) = \tau_i(b_1^i, \dots, b_{k_i}^i)) &\implies \\ \exists x : f(x) = \tau(\tau_1(a_1^1, \dots, a_{k_1}^1), \dots, \tau_n(a_1^n, \dots, a_{k_n}^n)) \wedge \\ \wedge g(x) = \tau(\tau_1(b_1^1, \dots, b_{k_1}^1), \dots, \tau_n(b_1^n, \dots, b_{k_n}^n)) \wedge \\ &\wedge \bigotimes_{i=1}^n \chi(x_i) \leq \chi(x) \end{aligned}$$

and combining with the same condition on the τ_i one obtains

$$\begin{aligned} \bigwedge_{i=1}^n \bigwedge_{j=1}^{k_i} (f(x_j^i) = a_j^i \wedge g(x_j^i) = b_j^i) &\implies \\ \exists x : f(x) = \tau(\tau_1(a_1^1, \dots, a_{k_1}^1), \dots, \tau_n(a_1^n, \dots, a_{k_n}^n)) \wedge \\ \wedge g(x) = \tau(\tau_1(b_1^1, \dots, b_{k_1}^1), \dots, \tau_n(b_1^n, \dots, b_{k_n}^n)) \wedge \\ &\wedge \bigotimes_{i=1}^n \bigotimes_{j=1}^{k_i} \chi(x_j^i) \leq \chi(x) \end{aligned}$$

This concludes the proof since every linear term can be written as a concatenation of operations.

For affine, relevant and cartesian terms the considerations done in the proof of proposition 8 can easily be adapted to the span case. \square

Theorem VIII.17. *Let \mathcal{E} be a topos and (Q, \otimes, k, \leq) an internal partially ordered commutative monoid. Let $i : (\Sigma_1, E_1) \rightarrow (\Sigma_2, E_2)$ be a linear interpretation of signatures. There is a strict monoidal functor*

$$i^* : \mathbf{Span}_{(\Sigma_2, E_2)}^{\text{lin}}(Q) \rightarrow \mathbf{Span}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)$$

The assignment $i \mapsto i^*$ extends to a contravariant functor.

Similar results hold for affine, relevant and cartesian interpretations and spans.

Proof. i^* is defined as in the relational case, sending algebras of type (Σ_2, E_2) to their interpretations of type (Σ_1, E_1) . Being it identity on morphisms by definition (the fact that morphisms of $\mathbf{Span}_{(\Sigma_2, E_2)}^{\text{lin}}(Q)$ are also morphisms of $\mathbf{Span}_{(\Sigma_1, E_1)}^{\text{lin}}(Q)$ is a direct consequence of proposition 13) functoriality follows trivially. \square

Theorem IX.2. *Let \mathcal{E}, \mathcal{F} be toposes, and $L : \mathcal{E} \rightarrow \mathcal{F}$ be a logical functor. Let (Q, \otimes, k, \vee) be an internal commutative quantale in \mathcal{E} and (Σ, E) be a signature. There is a symmetric monoidal functor*

$$L^* : \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)$$

The assignment $L \mapsto L^*$ is functorial.

Proof. The proof heavily relies on the fact that logical morphisms preserve models of logical theories: We know that, if T is a logical theory, a logical functor $L : \mathcal{E} \rightarrow \mathcal{F}$ preserves every interpretation (that is, every model), of T in \mathcal{E} . This is because an interpretation of T in \mathcal{E} assigns to every term and formula its correspondent in the Mitchell-Bénabou internal language: Every type is interpreted in a product of objects, and every constant into a morphism of \mathcal{E} . The axioms correspond, finally, to commutative diagrams in \mathcal{E} . Since these diagrams involve only limits, exponentials and subobject classifiers, they are preserved by L up to isomorphism. This means that the image through L of objects and morphisms that constitute a model of T in \mathcal{E} is a model of T in \mathcal{F} . The idea is then to state our definition of composition and identity of $\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q)$ in terms of logical theories: In this case the composition and the identity of two algebra-preserving relations will be just a model of this theory in \mathcal{E} , and will hence be preserved by L . The images through L of our relations will then still satisfy our definition of composition in the internal language of \mathcal{F} , guaranteeing that $L(R \circ S)(a, c) = \bigvee_b \{LR(a, b) \otimes LS(b, c)\} = (LR \circ LS)(a, c)$. From this, we can define $L^* : \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) \rightarrow \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)$ as follows:

- On objects, $L^*(A) = L(A)$
- On morphisms, denoting with κ the canonical isomorphism from $LA \times LB$ to $L(A \times B)$,

$$L^*(R) = LA \times LB \xrightarrow{\kappa} L(A \times B) \xrightarrow{LR} LQ$$

Now we have to state what our composition is in terms of logical theories. Given a signature (Σ, E) , we can define a logical theory

$$T = (A, B, C, Q, \{\sigma_i^A\}_{\sigma_i \in \Sigma}, \{\sigma_i^B\}_{\sigma_i \in \Sigma}, \{\sigma_i^C\}_{\sigma_i \in \Sigma}, \otimes, \vee, k, 1_B, R_{AB}, R_{BC}, R_{AC}),$$

where

- For a given $\sigma_i \in \Sigma$ of arity n_i ,
 - σ_i^A is a constant of type $A^{A^{n_i}}$
 - σ_i^B is a constant of type $B^{B^{n_i}}$
 - σ_i^C is a constant of type $C^{C^{n_i}}$

- \otimes is a constant of type $Q^{Q \times Q}$
- \vee is a constant of type Q^{PQ}
- k is a constant of type Q
- 1_B is a constant of type $Q^{B \times B}$
- R_{AB}, R_{BC}, R_{CD} are constants of type $Q^{A \times B}, Q^{B \times C}, Q^{A \times C}$, respectively

We require this theory to satisfy the set of axioms $\{\alpha_{1_B}, \{\alpha_A\}, \{\alpha_B\}, \{\alpha_C\}, \{\alpha_Q\}, \{\alpha_{R_{AB}}\}, \{\alpha_{R_{BC}}\}, \alpha_{comp}\}$, where:

- $\{\alpha_A\}$ is the set of axioms that makes $(A, \{\sigma_i^A\}_{\sigma_i \in \Sigma})$ into an algebra of type (Σ, E)
- $\{\alpha_B\}$ is the set of axioms that makes $(B, \{\sigma_i^B\}_{\sigma_i \in \Sigma})$ into an algebra of type (Σ, E)
- $\{\alpha_C\}$ is the set of axioms that makes $(C, \{\sigma_i^C\}_{\sigma_i \in \Sigma})$ into an algebra of type (Σ, E)
- $\{\alpha_Q\}$ is the set of axioms that makes (Q, \otimes, k, \vee) into a quantale
- α_{1_B} is the axiom $\forall b, b'. 1_B(b, b') = \bigvee \{k|b = b'\}$
- $\{\alpha_{R_{AB}}\}$ is the set of all the axioms, one for every $\sigma_i \in \Sigma$ of arity n , of the form

$$\begin{aligned} & \forall_{a_1, \dots, a_n, b_1, \dots, b_n} \bigvee \left\{ R_{AB}(\sigma_i^A(a_1, \dots, a_n), \sigma_i^B(b_1, \dots, b_n)), \bigotimes_{j=1}^n R(a_j, b_j) \right\} = \\ & = R_{AB}(\sigma_i^A(a_1, \dots, a_n), \sigma_i^B(b_1, \dots, b_n)) \end{aligned}$$

(Note that in this setting to use the quantale order relation we have to write explicitly what it is. The axiom above is nothing but the algebra preservation axiom for R_{AB} written explicitly using the algebraic lattice structure)

- $\{\alpha_{R_{BC}}\}$ is the set of all the axioms, one for every $\sigma_i \in \Sigma$ of arity n , of the form

$$\begin{aligned} & \forall_{b_1, \dots, b_n, c_1, \dots, c_n} \bigvee \left\{ R_{BC}(\sigma_i^B(b_1, \dots, b_n), \sigma_i^C(c_1, \dots, c_n)), \bigotimes_{j=1}^n R(b_j, c_j) \right\} = \\ & = R_{BC}(\sigma_i^B(b_1, \dots, b_n), \sigma_i^C(c_1, \dots, c_n)) \end{aligned}$$

- Finally, α_{comp} is the axiom

$$\forall a, c. R_{AC}(a, c) = \bigvee \{R_{AB}(a, b) \otimes R_{BC}(b, c) | b \in B\}$$

An interpretation of this theory in the topos \mathcal{E} then consists of three morphisms in \mathcal{E}

$$R_{AB} : A \times B \rightarrow Q \quad R_{BC} : B \times C \rightarrow Q \quad R_{AC} : A \times C \rightarrow Q$$

Where the sets of axioms $\{\alpha_A\}, \{\alpha_B\}, \{\alpha_C\}$ get interpreted into commutative diagrams ensuring that A, B, C are internal algebras of signature (Σ, E) , respectively, while $\{\alpha_{R_{AB}}\}, \{\alpha_{R_{BC}}\}$ guarantee that R_{AB} and R_{BC} respect the usual algebraic condition. $\{\alpha_Q\}$ gets interpreted into diagrams ensuring that Q is an internal quantale and α_{comp} guarantees

that R_{AC} is exactly the composition of relations R_{AB}, R_{BC} in $\mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q)$. \square

Proposition 3. *With the same assumptions, the induced functor L^* of theorem IX.2 commutes with graphs and converses. That is, the following diagrams commute:*

$$\begin{array}{ccc} \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) & \xrightarrow{L^*} & \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ) \\ \uparrow (-)_\circ & & \uparrow (-)_\circ \\ \mathbf{Alg}^{\mathcal{E}}(\Sigma, E) & \xrightarrow{L} & \mathbf{Alg}^{\mathcal{F}}(\Sigma, E) \\ \\ \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q)^{op} & \xrightarrow{(L^*)^{op}} & \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ)^{op} \\ \downarrow (-)^\circ & & \downarrow (-)^\circ \\ \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{E}}(Q) & \xrightarrow{L^*} & \mathbf{Rel}_{(\Sigma, E)}^{\mathcal{F}}(LQ) \end{array}$$

Proof. Start noting that the graph functor is identity on objects, so trivially $L^*(A)_\circ = L^*A = (L^*A)_\circ$ for every object A . For a morphism $f : A \rightarrow B$, in \mathcal{E} , consider the diagram:

$$\begin{array}{ccccc} LA \times LB & \xrightarrow{\text{iso}} & L(A \times B) & \xrightarrow{L(f \times 1_B)} & L(B \times B) & \xrightarrow{L(\text{id}_B^{\text{Rel}})} & L(Q) \\ & \searrow & \downarrow \text{iso} & & \downarrow \text{iso} & \nearrow \text{id}_{LB}^{\text{Rel}} & \\ & & LA \times LB & \xrightarrow{Lf \times L1_B} & LB \times LB & & \end{array}$$

Where id_B^{Rel} is the morphism of \mathcal{E} that defines 1_B in $\mathbf{Rel}_{\Sigma, E}^{\mathcal{E}}(Q)$. The top row of the diagram is just $L^*(f)_\circ$, while the bottom one is $(L^*f)_\circ$. The left triangle commutes trivially, the center square commutes because L preserves products, the right triangle commutes because L preserves relational identities (previous proposition). Preservation of the converse follows trivially from the fact that any logical functor preserves products. \square

Theorem IX.4. *Let \mathcal{E}, \mathcal{F} be toposes, and $L : \mathcal{E} \rightarrow \mathcal{F}$ be a logical functor. Let (Q, \otimes, k, \leq) be an internal partially ordered commutative monoid in \mathcal{E} and (Σ, E) be a signature. There is a symmetric monoidal functor*

$$L^* : \mathbf{Span}_{(\Sigma, E)}^{\mathcal{E}}(Q) \rightarrow \mathbf{Span}_{(\Sigma, E)}^{\mathcal{F}}(LQ)$$

The assignment $L \mapsto L^$ is functorial.*

Proof. Here the same considerations used to prove theorem IX.2 hold. Given a signature (Σ, E) , the logical theory we use is

$$T = (X, A, B, Q, \{\sigma_i^A\}_{\sigma_i \in \Sigma}, \{\sigma_i^B\}_{\sigma_i \in \Sigma}, f, g, \chi, \otimes, \leq, k),$$

where

- For a given $\sigma_i \in \Sigma$ of arity n_i ,
 - σ_i^A is a constant of type $A^{A^{n_i}}$
 - σ_i^B is a constant of type $B^{B^{n_i}}$
 - σ_i^C is a constant of type $C^{C^{n_i}}$

- f, g, χ are constants of type X^A, X^B, X^Q , respectively
- \otimes is a constant of type $Q^{Q \times Q}$
- \leq is a constant of type $\Omega^{Q \times Q}$
- k is a constant of type Q

We require this theory to satisfy the set of axioms $\{\{\alpha_A\}, \{\alpha_B\}, \{\alpha_Q\}, \{\alpha_X\}\}$, where:

- $\{\alpha_A\}$ is the set of axioms that makes $(A, \{\sigma_i^A\}_{\sigma_i \in \Sigma})$ into an algebra of type (Σ, E)
- $\{\alpha_B\}$ is the set of axioms that makes $(B, \{\sigma_i^B\}_{\sigma_i \in \Sigma})$ into an algebra of type (Σ, E)
- $\{\alpha_C\}$ is the set of axioms that makes $(C, \{\sigma_i^C\}_{\sigma_i \in \Sigma})$ into an algebra of type (Σ, E)
- $\{\alpha_Q\}$ is the set of axioms that makes (Q, \otimes, k, \leq) into a partially ordered monoid
- $\{\alpha_X\}$ is the set of axioms, one for every $\sigma \in \Sigma$ of arity n , of the form

$$\begin{aligned} \forall x_1, \dots, x_n \exists x. f(x) &= \sigma_i^A(f(x_1), \dots, f(x_n)) \wedge \\ &\wedge (g(x) = \sigma_i^B(g(x_1), \dots, g(x_n))) \wedge \\ &\wedge \bigotimes_{j=1}^n \chi(x_j) \leq \chi(x) \end{aligned}$$

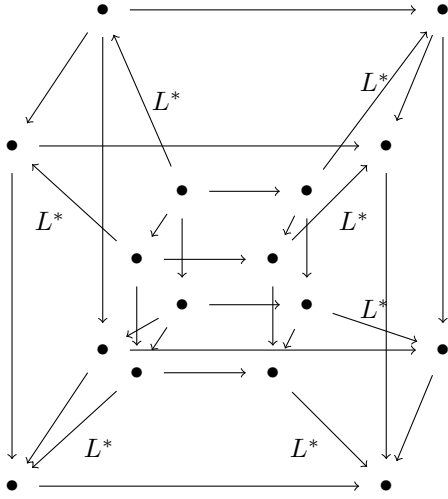
A model of T in \mathcal{E} is just a span that respects the algebraic structure and we know that L preserves this condition. L^* then agrees with L on objects and is defined as $(LX, Lf, Lg, L\chi)$ on the morphism (X, f, g, χ) . For composition and identity we do not need to invoke any logical theory: The identity span is of the form $(X, 1_X, 1_X, \chi_k)$, and the span part is clearly preserved because functors preserve identities in general. The quantale part χ_k is the morphism $A \rightarrow 1 \rightarrow Q$ where the final arrow sends the terminal object to the quantale unit. Again, being L logical this is trivially preserved. For composition, note that the span part is composed via pullbacks and L preserves limits. For the quantale part we have, supposing (Z, h, k, ζ) to be the composite of (X, f, g, χ) and (Y, f', g', ν) ,

$$\begin{array}{ccccccc} LZ & \xrightarrow{L\langle p_1, p_2 \rangle} & L(X \times Y) & \xrightarrow{L(\chi \times \nu)} & L(Q \times Q) & \xrightarrow{L(\otimes)} & L(Q) \\ & \searrow & \downarrow \text{iso} & & \downarrow \text{iso} & \nearrow \otimes & \\ & & LX \times LY & \xrightarrow{L\chi \times L\nu} & LQ \times LQ & & \end{array}$$

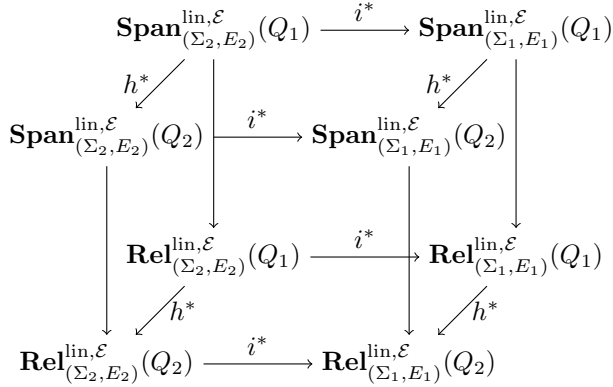
Where p_1, p_2 are the pullback projections. The top row is the image of ζ through L . The triangle on the left and the square on the center commute because L preserves limits, while the triangle on the right commutes because every partially ordered monoid is obviously a model of a theory, so the multiplication of Q gets carried in the multiplication of LQ . \square

Theorem X.1. *Let \mathcal{E} be a topos, $h : Q_1 \rightarrow Q_2$ a morphism of internal commutative quantales, $i : (\Sigma_1, E_1) \rightarrow (\Sigma_2, E_2)$ a linear interpretation and $L : \mathcal{E} \rightarrow \mathcal{F}$ a logical functor. For the induced functors of theorems VII.1, VII.3, VIII.10, VIII.17, IX.2 and IX.4, the following diagram commutes (be aware that in the hypercube below commutative squares involving L^* only*

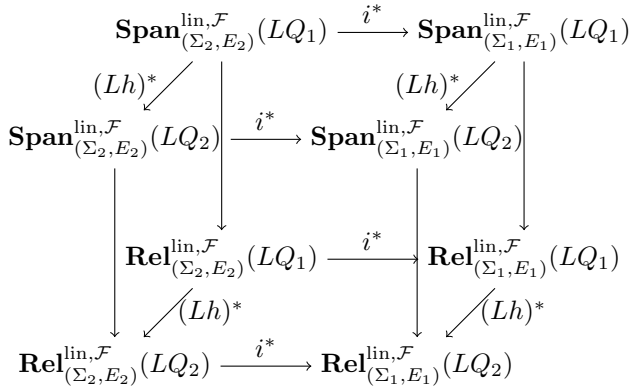
commute up to isomorphism. Other squares commute up to equality):



Where the inner cube is



and the outer cube is



In both cases the vertical arrows are the functors of theorem VI.1. Similar diagrams commute for affine, relevant and cartesian interpretations, relations and spans.

Proof. Here the notation $A \simeq B$ will denote that A and B are isomorphic. i^* trivially commutes with h^* , since the first is identity on morphisms and the second is identity on objects; for the very same reason, i^* commutes with V . V commutes

with h^* because the former acts by postcomposition with a homomorphism of quantales, that commutes with joins and orders.

To show that $L^*i^* \simeq i^*L^*$, note that for morphisms this is trivial, being i^* the identity on them. Let then $\langle\langle A, \sigma_j^A \rangle\rangle$ be an object of, say, $\mathbf{Rel}_{(\Sigma_2, E_2)}^E(Q)$, and consider $L^*i^*\langle\langle A, \sigma_j^A \rangle\rangle$. By definition this is equal to $L^*\langle\langle A, i(\sigma^A)_{j'} \rangle\rangle$, where every $i(\sigma^A)_{j'}$ is a term derived from the σ_j^A , so a composition of σ_j^A (and eventually diagonals and projections, depending on the interpretation). Being L logical, operations of A get carried into operations of LA , hence $L^*\langle\langle A, i(\sigma^A)_{j'} \rangle\rangle = \langle\langle LA, Li(\sigma^A)_{j'} \rangle\rangle$ is an algebra of type (Σ_1, E_1) in $\mathbf{Rel}_{(\Sigma_1, E_1)}^F(LQ)$. But, being $i(\sigma^A)_{j'}$ a composition of operations, projections and diagonals, and being L product preserving, it is $Li(\sigma^A)_{j'} \simeq i(\sigma^{LA})_{j'}$. Hence

$$\begin{aligned} L^*i^*\langle\langle A, \sigma_j^A \rangle\rangle &= \langle\langle LA, Li(\sigma^A)_{j'} \rangle\rangle \\ &\simeq \langle\langle LA, i(\sigma^{LA})_{j'} \rangle\rangle \\ &= i^*L^*\langle\langle A, \sigma_j^A \rangle\rangle \end{aligned}$$

The proof is the same when L^* and i^* act on spans.

To prove that $L^*V \simeq V^*L^*$, consider the following logical theory:

$$T = (X, A, B, Q, \{\sigma_i^A\}_{\sigma_i \in \Sigma}, \{\sigma_i^B\}_{\sigma_i \in \Sigma}, f, g, \chi, \otimes, \vee, k, R)$$

Where:

- $(X, A, B, Q, \{\sigma_i^A\}_{\sigma_i \in \Sigma}, \{\sigma_i^B\}_{\sigma_i \in \Sigma}, f, g, \chi, \otimes, \vee, k)$ is the fragment that states that (Q, \otimes, k, \vee) is a quantale, that $\langle\langle A, \{\sigma_i^A\}_{\sigma_i \in \Sigma} \rangle\rangle$ and $\langle\langle B, \{\sigma_i^B\}_{\sigma_i \in \Sigma} \rangle\rangle$ are algebras of the required signature and that (X, f, g, χ) is an algebraic preserving span over Q , with all the obvious axioms required to hold (see proof of theorems IX.2 and IX.4 for details)
- R is a constant of type $Q^{A \times B}$ together with the axioms that say it is an algebraic preserving relation over Q (again refer to the relational case in theorem IX.2)
- The additional axiom

$$R(a, b) = \bigvee \{ \chi(m) \mid \exists m. (s(m) = a \wedge t(m) = b) \}$$

is satisfied.

This logical theory expresses exactly the fact that R is a relation coming from a span in the sense of the order functor, so if $R = V(X, s, t, \chi)$ then R and (X, s, t, χ) are a model for T . From this we get an isomorphism between LR and $V(LX, Lf, Lg, L\chi)$, and hence

$$L^*V(X, f, g, \chi) = L^*R \simeq LR \simeq V(LX, Lf, Lg, L\chi)$$

Finally, to verify that $L^*h^* \simeq h^*L^*$, just note that it is possible to state what a quantale homomorphism is in terms of logical theories. This guarantees that if $h : Q_1 \rightarrow Q_2$ is a homomorphism of quantales, so is Lh . Everything then follows from the fact that h^* acts by postcomposition and L respects it. \square